

FRAMED CORD ALGEBRA INVARIANT OF KNOTS IN $S^1 \times S^2$

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ABSTRACT. We generalize Ng's two-variable algebraic/combinatorial 0-th framed knot contact homology for framed oriented knots in S^3 to knots in $S^1 \times S^2$, and prove that the resulting knot invariant is the same as the framed cord algebra of knots. Actually, our cord algebra has an extra variable, which potentially corresponds to the third variable in Ng's three-variable knot contact homology. Our main tool is Lin's generalization of the Markov theorem for braids in S^3 to braids in $S^1 \times S^2$. We conjecture that our framed cord algebras are always finitely generated for non-local knots.

1. INTRODUCTION

The dream of finding new higher categorical quantum invariants of smooth 4-manifolds that can distinguish smooth structures beyond Donaldson/Seiberg-Witten/Heegaard-Floer theory is largely unrealized, despite the spectacular success for new invariants in 3-dimensions and recent progress in higher category theory. A potentially new quantum invariant would be to promote the relative knot contact homology of knots in S^3 in [8] to a $(3+1)$ -TQFT-type theory (presumably the 0-th part of the BRST cohomology of a topological string theory). One lesson from $(2+1)$ -dimensions is the emergence of powerful diagrammatical techniques as exemplified by the Kauffman bracket definition of the Jones polynomial, and the subsequently elementary formulation of Turaev-Viro and Reshetikhin-Turaev $(2+1)$ -TQFTs. We see a striking parallel between the cord algebra invariant and the Jones polynomial.

In [8], the 0-th part of the relative knot contact homology in S^3 is interpreted using cords and skein relations—the main ingredients of diagrammatical techniques in $(2+1)$ -dimensions, analogous to the reformulation of the Jones polynomial of knots from von Neumann algebra

Key words and phrases. knot, braid group, knot contact homology, cord algebra.

The authors are partially supported by NSF DMS 1108736. Both authors thank Lenny Ng for valuable comments on the paper. He pointed out that the cord algebras of local knots are infinitely generated, and that the ends of the cords should stay on the framing.

using knot diagrams and the Kauffman bracket. Taking the elementary framed cord algebra invariant of knots in general 3-manifolds M as the main object of interest, we will follow the diagrammatical approach to constructing $(2 + 1)$ -TQFTs such as the Turaev-Viro and Reshetikhin-Turaev TQFTs. As a first step, we generalize Ng's two-variable combinatorial/algebraic 0-th framed knot contact homology for framed oriented knots in S^3 to knots in $S^1 \times S^2$, and prove that the resulting knot invariant is the same as the framed cord algebra of knots. Actually, our cord algebra has an extra variable, which potentially corresponds to the third variable in Ng's three-variable knot contact homology [9].

It is conjectured in [7] that the cord algebra invariant of knots in a general 3-manifold M is the 0-th relative knot contact homology. We do not prove this conjecture and will not use any knot contact homology theory. Instead we provide an algebraic version of this conjectured 0-th knot contact homology for knots in $S^1 \times S^2$ following [8] and regard our algebraic definition of the cord algebra as an effective method to calculate the topologically defined cord algebra invariant of knots. Our long term goal is to understand the higher categories underlying this algebraic formulation with an eye towards to a diagram construction of a $(3 + 1)$ -TQFT-type theory.

A second reason for our interest in the framed cord algebra invariant of knots is the conjectured relation between the augmentation polynomial and the Homfly polynomial of knots. A well-known question since the discovery of the Jones polynomial is how to place the Jones polynomial within classical topology (since knots are determined by their complements, so any knot invariant is determined by the homeomorphism type of the knot complement). The cord algebra of a knot is basically within classical topology, so the establishment of the conjectured relation between the augmentation polynomial and the Homfly polynomial is one answer to an old question.

To generalize the algebraic 0-th knot contact homology in [8] from S^3 to $S^1 \times S^2$, we use Lin's generalization of the Markov theorem for braids in S^3 to braids in $S^1 \times S^2$ [5] developed for defining a Jones polynomial of knots in $S^1 \times S^2$.¹

The rest of the paper is organized as follows. In Section 2.1, we introduce the Markov theorem for knots in $S^1 \times S^2$, which are represented by the closure of elements in \mathcal{C}_n , the Artin group with Dynkin diagram

¹This generalization, eventually rendered unnecessary for the intended application by Witten's work, finds a similar application in our work. We dedicate our work to X.-S. Lin—an important vanguard in quantum knot theory.

B_n . In Section 2.2, we give several actions of \mathcal{C}_n on free algebras. We interpret these actions both algebraically and topologically. These actions will be the key ingredients to define the invariant HC_0 in Section 3.1. In Section 3.2 - Section 3.4, we compute some specific examples, demonstrate some useful propositions, and prove the invariance of HC_0 under Markov moves, respectively. Section 4.1 - Section 4.4 are devoted to prove several properties of the HC_0 invariant. We study two special classes of knots in $S^1 \times S^2$, torus knots and local knots. Moreover, we derive a family of invariants, called augmentations, from HC_0 . Finally, in Section 5 we prove that the HC_0 invariant has a nice topological interpretation as the framed cord algebra defined in [8].

The first author also created a Mathematica package for computer calculations of the HC_0 invariant and augmentation numbers. The program can be found at [10] and is partly motivated by Ng's computer package, which was used to compute various invariants derived from knot contact homology for knots in S^3 . To run the program, one needs to install the non-commutative algebra package NCAAlgebra/NCGB [4].

2. MARKOV MOVES AND ACTIONS OF \mathcal{C}_n ON FREE ALGEBRAS

First we provide some background materials. Links and knots in this paper are always framed and oriented.

2.1. Markov moves in $S^1 \times S^2$. In this subsection, we describe a theorem on Markov moves for links in $S^1 \times S^2$. See [5] for a more detailed discussion.

The classical braid group with n strands, \mathcal{B}_n , is defined by the presentation $\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2 \rangle$. It is the Artin group with Dynkin diagram type A_{n-1} , and can also be viewed as the braid group on the 2-disk $D^2 \subset \mathbb{R}^2$.

Any link in S^3 can be represented as the closure of some braid in the classical braid group. The Markov theorem states that two braids B, B' give rise to the same link if and only if B' can be obtained from B by a finite sequence of the following operations or their inverses:

- 1). change $B \in \mathcal{B}_n$ to one of its conjugates in \mathcal{B}_n ;
- 2). change $B \in \mathcal{B}_n$ to $B\sigma_n^{\pm 1} \in \mathcal{B}_{n+1}$.

The Markov theorem for links in S^3 is generalized to links in $S^1 \times S^2$ in [5] as follows.

Let \mathcal{C}_n be the Artin group corresponding to the Dynkin diagram B_n generated by $\alpha_0, \dots, \alpha_{n-1}$, with the following generating relations:

- 1). $\alpha_i \alpha_j = \alpha_j \alpha_i, |i - j| \geq 2$
- 2). $\alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1}, i \geq 1$
- 3). $\alpha_0 \alpha_1 \alpha_0 \alpha_1 = \alpha_1 \alpha_0 \alpha_1 \alpha_0$.

Clearly, we have natural inclusions $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_n \subset \cdots$. We denote by ϵ^- these natural embeddings.

It is shown in [2] that \mathcal{C}_n is isomorphic to the braid group on the annulus $[0, 1] \times S^1$, or the 1-punctured disk. Specifically, the isomorphism is illustrated in Figure 1.

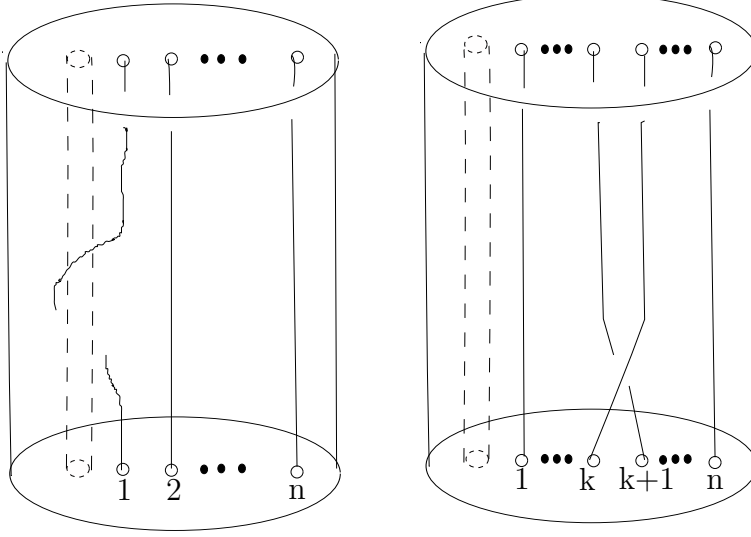


FIGURE 1. α_0 and $\alpha_k, k \geq 1$

Simply treating $\{\text{puncture}\} \times [0, 1]$ as the first strand of the new braid, we can regard a braid on the 1-punctured disk as a braid on the disk. Thus we have an embedding of \mathcal{C}_n into \mathcal{B}_{n+1} . Denote the generators of \mathcal{B}_{n+1} by $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$. Then the embedding from \mathcal{C}_n to \mathcal{B}_{n+1} is given by the following map:

$$\mathcal{C}_n \longrightarrow \mathcal{B}_{n+1}, \quad \alpha_0 \longmapsto \sigma_0^2, \quad \alpha_i \longmapsto \sigma_i, i \geq 1.$$

From now on, we will identify \mathcal{C}_n with its image in \mathcal{B}_{n+1} , which is the subgroup consisting of the braids that fix the first puncture.

The correspondence between braids on the annulus and links in $S^1 \times S^2$ is obtained via open book decompositions.

Consider the standard open book decomposition of S^3 with an unknot J as the binding. Let K be another unknot which is a closed braid with respect to the braid axis J . Then

$$M = \overline{S^3 \setminus (J \times D^2 \cup K \times D^2)}$$

is a fibration over S^1 whose fibre is an annulus $[0, 1] \times S^1$. $S^1 \times S^2$ is obtained by a 0-Dehn surgery along K . Thus $S^1 \times S^2 = M \sqcup_f D^2 \times S^1$, where f is the gluing homeomorphism which maps the meridian of the

solid torus to $K \times z_0, z_0 \in \partial D^2$. Let K^* be the image of $0 \times S^1$ under f in $S^1 \times S^2$, where $0 \times S^1$ is the core of the solid torus. We call K^* the dual knot of K . Then the fibration on M extends to an open book decomposition on $S^1 \times S^2$ with the binding $J \cup K^*$. Note that $S^1 \times S^2 \setminus (J \cup K^*)$ is homeomorphic to the product of the annulus with S^1 . It's not hard to see that any link in $S^1 \times S^2$ can be isotoped into $S^1 \times S^2 \setminus (J \cup K^*)$ transversal to each page, and thus becomes a braid on the annulus.

To state the Markov theorem, we need one more lemma.

Define a map $\epsilon^+ : \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$,

$$(2.1) \quad \epsilon^+(\alpha_i) = \begin{cases} \alpha_1 \alpha_0 \alpha_1 & i = 0 \\ \alpha_{i+1} & i \geq 1 \end{cases}$$

The map ϵ^+ has a nice geometrical interpretation if we view \mathcal{C}_n as the braid group on the annulus. The map simply inserts a straight strand right next to the line $\{\text{puncture}\} \times [0, 1]$. See Figure 2.

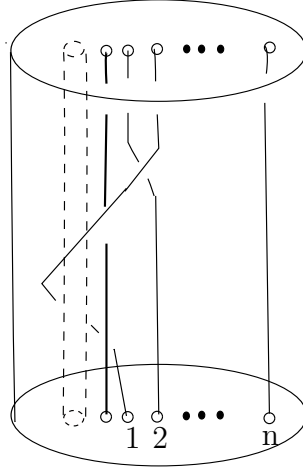


FIGURE 2. $\epsilon^+(\alpha_1 \alpha_0)$

Note that the newly inserted line will be labeled by 1, and the other strands' labels will be shifted up by 1.

Lemma 1. [5] The map ϵ^+ is an injective group homomorphism.

Proof From the geometrical interpretation of the map, it should be clear that it is an injective group homomorphism. For a rigorous algebraic proof, see [5]. \square

Remark 1. Now there are two embeddings of \mathcal{C}_n into \mathcal{C}_{n+1} , namely the natural inclusion ϵ^- and the map ϵ^+ . From the geometric point of view, ϵ^- is to place a strand on the far right of the braid, while ϵ^+ is to insert a strand right next to the line $\{\text{puncture}\} \times [0, 1]$.

Here is the statement of the Markov Theorem for links in $S^1 \times S^2$.

Theorem 1. [5] The closures of two braids $\beta, \beta' \in \cup_{n=1}^{\infty} \mathcal{C}_n$ give the same link in $S^1 \times S^2$ if and only if there is a finite sequence of braids, $\beta = \beta_0, \beta_1, \dots, \beta_k = \beta'$, such that β_{i+1} can be obtained from β_i by one of the following operations or their inverses:

- 1). change $\beta_i \in \mathcal{C}_n$ to one of its conjugates in \mathcal{C}_n ;
- 2). change $\beta_i \in \mathcal{C}_n$ to $\epsilon^-(\beta_i)\alpha_n^{\pm} \in \mathcal{C}_{n+1}$;
- 3). change $\beta_i \in \mathcal{C}_n$ to $\epsilon^+(\beta_i)\alpha_1^{\pm} \in \mathcal{C}_{n+1}$.

Remark 2. Given a braid $\beta \in \mathcal{C}_n$, we can obtain the knot in $S^1 \times S^2$ represented by β as follows. Take a punctured disk $D' = D \setminus B_{\epsilon}(0)$, and let $X = D' \times [0, 1]$. Draw the diagram of β inside X . Then $S^1 \times S^2$ is obtained by identifying the top and the bottom punctured disk and then gluing a solid torus to each torus boundary component. The gluing maps are given by sending the meridian of each solid torus to $z_0 \times S^1$ and $z_1 \times S^1$, respectively for some z_0 on the boundary of the puncture and z_1 on the outer boundary of D' . And the knot represented by β is the image of the braid diagram in $S^1 \times S^2$. See Figure 3 for $\beta = \alpha_0\alpha_1$.

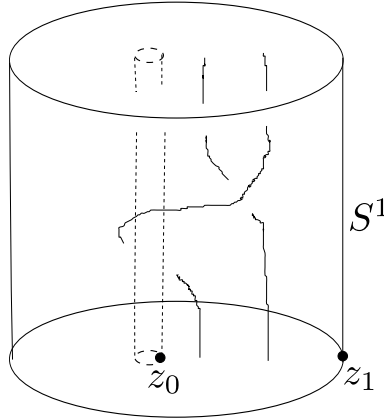


FIGURE 3. The closure of β in $S^1 \times S^2$

2.2. Actions of \mathcal{C}_n on free algebras. Throughout the paper, R denotes the commutative ring $\mathbb{Z}[\lambda^{\pm}, \mu^{\pm}, \Gamma^{\pm}]$. We define several free non-commutative algebras over the ring R as follows.

$$\mathcal{A}_n^+ := R\langle a_{ij}^x, 0 \leq i, j \leq n, x \in \mathbb{Z} \rangle / \langle a_{ii}^0 - (1 + \mu)\Gamma, 0 \leq i \leq n \rangle,$$

$$\begin{aligned}\mathcal{A}_n^- &:= R\langle a_{ij}^x, 1 \leq i, j \leq n+1, x \in \mathbb{Z} \rangle / \langle a_{ii}^0 - (1+\mu)\Gamma, 1 \leq i \leq n+1 \rangle, \\ \mathcal{A}_n &:= R\langle a_{ij}^x, 1 \leq i, j \leq n, x \in \mathbb{Z} \rangle / \langle a_{ii}^0 - (1+\mu)\Gamma, 1 \leq i \leq n \rangle.\end{aligned}$$

The algebra \mathcal{A}_n can be embedded into \mathcal{A}_n^+ and \mathcal{A}_n^- in the most natural way. We will always identify \mathcal{A}_n with its images in \mathcal{A}_n^+ and \mathcal{A}_n^- .

Now we introduce an action of \mathcal{C}_n on \mathcal{A}_n , and extend the action to the larger algebras $\mathcal{A}_n^+, \mathcal{A}_n^-$. The action is first presented algebraically and then will be given a topological interpretation.

Recall that the generators \mathcal{C}_n are denoted by $\alpha_0, \dots, \alpha_{n-1}$, which satisfy the relation given in Section 2.1. We define a group morphism $\Phi : \mathcal{C}_n \longrightarrow \text{Aut}(\mathcal{A}_n)$ as follows.

For $1 \leq k \leq n-1$,

$$(2.2) \quad \Phi(\alpha_k)(a_{ij}^x) = \begin{cases} -a_{k+1,j}^x + \frac{1}{\Gamma\mu}a_{k+1,k}^0a_{k,j}^x & i = k, j \neq k, k+1 \\ -a_{k+1,k}^x + \frac{1}{\Gamma\mu}a_{k+1,k}^0a_{k,k}^x & i = k, j = k+1 \\ a_{k+1,k+1}^x - \frac{1}{\Gamma}a_{k+1,k}^xa_{k,k+1}^0 - \frac{1}{\Gamma\mu}a_{k+1,k}^0a_{k,k+1}^x + \frac{1}{\Gamma^2\mu}a_{k+1,k}^0a_{k,k}^xa_{k,k+1}^0 & i = k, j = k \\ a_{k,j}^x & i = k+1, j \neq k, k+1 \\ -a_{k,k+1}^x + \frac{1}{\Gamma}a_{k,k}^xa_{k,k+1}^0 & i = k+1, j = k \\ a_{k,k}^x & i = k+1, j = k+1 \\ -a_{i,k+1}^x + \frac{1}{\Gamma}a_{i,k}^xa_{k,k+1}^0 & i \neq k, k+1, j = k \\ a_{i,k}^x & i \neq k, k+1, j = k+1 \\ a_{i,j}^x & i \neq k, k+1, j \neq k, k+1 \end{cases}$$

$$(2.3) \quad \Phi(\alpha_0)(a_{ij}^x) = \begin{cases} a_{1,1}^x & i = 1, j = 1 \\ -\mu a_{1,j}^{x-1} + \frac{1}{\Gamma}a_{1,1}^xa_{1,j}^{-1} & i = 1, j \geq 2 \\ \frac{1}{\mu}(-a_{i,1}^{x+1} + \frac{1}{\Gamma}a_{i,1}^1a_{1,1}^x) & i \geq 2, j = 1 \\ a_{i,j}^x - \frac{1}{\Gamma\mu}a_{i,1}^{x+1}a_{1,j}^{-1} - \frac{1}{\Gamma}a_{i,1}^1a_{1,j}^{x-1} + \frac{1}{\Gamma^2\mu}a_{i,1}^1a_{1,1}^xa_{1,j}^{-1} & i \geq 2, j \geq 2 \end{cases}$$

It is not hard, though tedious, to check that Φ is well defined, i.e. $\Phi(\alpha_i)$ satisfies the braid relations that define \mathcal{C}_n .

We extend the action of \mathcal{C}_n to the algebra \mathcal{A}_n^+ by furthermore defining the action on $a_{0j}^x, a_{i0}^x, 0 \leq i, j \leq n$. This extended action will be denoted by Φ^+ .

$$(2.4) \quad \Phi^+(\alpha_0)(a_{ij}^x) = \begin{cases} a_{0,0}^x & i=0, j=0 \\ \frac{1}{\mu} a_{0,1}^{x+1} & i=0, j=1 \\ -a_{0,j}^x + \frac{1}{\Gamma\mu} a_{0,1}^{x+1} a_{1,j}^{-1} & i=0, j \geq 2 \\ \mu a_{1,0}^{x-1} & i=1, j=0 \\ -a_{i,0}^x + \frac{1}{\Gamma} a_{i,1}^1 a_{1,0}^{x-1} & i \geq 2, j=0 \end{cases}$$

For $1 \leq k \leq n-1$, $\Phi^+(\alpha_k)(a_{ij}^x)$ are given by the same equation as 2.2, except that now i, j are allowed to be zero when they are not k or $k+1$.

Similarly, the extended action of \mathcal{C}_n on \mathcal{A}_n^- is defined by Equations 2.2, 2.3 except that the range of i, j now is from 1 to $n+1$. We denote this action by Φ^- .

Again, it can be checked Φ^+, Φ^- are both well defined.

A few remarks are in order.

Remark 3. 1). From now on, for a braid $\beta \in C_n$, we will write $\Phi_\beta, \Phi_\beta^+, \Phi_\beta^-$ for $\Phi(\beta), \Phi^+(\beta), \Phi^-(\beta)$, respectively.

2). It's direct from the very definitions that $\Phi_\beta = (\Phi_\beta^+)_{|\mathcal{A}_n} = (\Phi_\beta^-)_{|\mathcal{A}_n}$. It's also clear that $\Phi_\beta^- = \Phi_{\epsilon^-(\beta)}$ if we identify \mathcal{A}_n^- with \mathcal{A}_{n+1} in the obvious way.

3). From the definition of \mathcal{C}_n in Section 2.1, it's easy to see that the subgroup generated by $\{\alpha_1, \dots, \alpha_{n-1}\}$ is isomorphic to the classical braid group on n strands. We denote this subgroup by \mathcal{B}_n . In Equation 2.2, if we set $\Gamma = -1, \mu = 1$, and $x = 0$, then $\Phi_{|\mathcal{B}_n}$ acting on $\mathbb{Z}\langle a_{ij}^0 \rangle$ is exactly the braid group action given in [6]. So our braid group action is a generalization of Ng's in [6].

The above actions will be less mysterious after we give a topological interpretation.

Let D be the unit disk in the complex plane centered at the origin, D_n be the punctured disk with $n+1$ punctures labeled, from left to right, by p, p_1, \dots, p_n and let $q_i = p_i - \epsilon, 1 \leq i \leq n, \epsilon > 0$ be n points in D_n close to the punctures. See Figure 4.

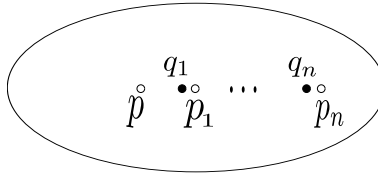


FIGURE 4. D_n

Let $Q_n = \{q_i, 1 \leq i \leq n\}$ and let $\mathcal{Q}_n = \{\gamma : [0, 1] \rightarrow D_n \mid \gamma \text{ is continuous, } \gamma(0), \gamma(1) \in Q_n\} / \sim$. Here \sim is the equivalence relation which means two curves $\gamma_1 \sim \gamma_2$ if and only if γ_1 and γ_2 are homotopic inside D_n relative to their end points. So the curves are not allowed to pass through any of the punctures and their end points are fixed during the homotopy. Then \mathcal{Q}_n is the set of equivalence classes of such curves.

Let $\tilde{\mathcal{A}}_n$ be the free non-commutative algebra over R generated by elements of \mathcal{Q}_n modulo the “skein” relations shown in Figure 5. Note that \otimes in Figure 5, and all other places of the paper, means the multiplication. And the second relation, as well as other similar relations in the context, depicts some local neighborhood of the diagrams outside of which they all agree.

$$\begin{aligned}
 1). \quad & \text{Diagram: a loop starting and ending at } q_i \text{ with a dot on the loop} = (1 + \mu)\Gamma \\
 2). \quad & \text{Diagram: a horizontal line with a dot at } q_i \text{ and a loop above it} + \text{Diagram: a horizontal line with a dot at } q_i \text{ and a loop below it} \\
 & = \frac{1}{\Gamma} \text{Diagram: a horizontal line with a dot at } q_i \text{ and a loop above it} \otimes \text{Diagram: a horizontal line with a dot at } q_i \text{ and a loop above it}
 \end{aligned}$$

FIGURE 5. skein relation

For $1 \leq i, j \leq n, x \in \mathbb{Z}$, let γ_{ij}^x and γ_i be the curves shown in Figure 6, namely γ_{ij}^x starts from q_i , winds around p counter clock-wise x times if $x \geq 0$, or clock-wise $-x$ times if $x < 0$, and finally goes through the upper half disk to reach q_j . γ_i is the curve that starts and ends at q_i and winds around p_i counter clock-wise once.

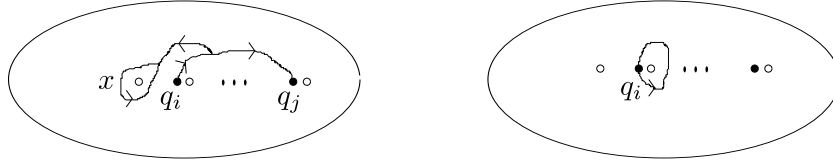


FIGURE 6. γ_{ij} and γ_i

It should be noted that the relations shown in Figure 7 can be derived from the ones in Figure 5. And the second relation in Figure 7 is equivalent to the property that if $\gamma, \gamma' \in \mathcal{Q}_n$ such that $\gamma(0) = q_i$ and $\gamma'(1) = q_i$, then $\gamma_i * \gamma = \mu\gamma, \gamma' * \bar{\gamma}_i = \mu^{-1}\gamma'$, where $*$ means connecting the two adjacent curves, and $\bar{\gamma}_i$ is the curve γ_i with reversed direction.

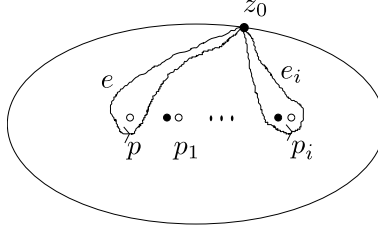
It's not hard to check that any curve can be decomposed into a (non-commutative) polynomial in γ_{ij}^x 's by repeated applications of

$$\begin{aligned}
1). \quad & \text{Diagram 1} + \text{Diagram 2} = \frac{1}{\Gamma\mu} \text{Diagram 3} \otimes \text{Diagram 4} \\
2). \quad & \text{Diagram 5} = \mu \text{Diagram 6} \quad \text{Diagram 7} = \mu^{-1} \text{Diagram 8}
\end{aligned}$$

FIGURE 7. Derived skein relations

the “skein” relations. Therefore $\tilde{\mathcal{A}}_n$ is generated by γ_{ij}^x ’s. Actually they turn out to be free generators after we construct an isomorphism between $\tilde{\mathcal{A}}_n$ and \mathcal{A}_n below. Of-course, since $\gamma_{ii}^0 = (1 + \mu)\Gamma$, this doesn’t count as part of the free generators.

Now pick a base point on the boundary of the disk D . To make it explicit, let us pick some z_0 on the upper half of the boundary as the base point. The fundamental group of D_n is the free group F_{n+1} on $n + 1$ generators, which we denote by e, e_1, \dots, e_n , where e_i is the loop that winds around p_i counter clock-wise once and e is the loop that winds around p counter clock-wise once. See Figure 8.

FIGURE 8. e and e_i

Firstly, we define an intermediate non-commutative algebra $\mathcal{B} = R\langle e^{\pm 1}, y_1, y_2, \dots, y_n \rangle / \mathcal{I}$, where \mathcal{I} is the two-sided ideal generated by $ee^{-1} - 1$, $e^{-1}e - 1$ and $y_i^2 - \Gamma(1 + \mu)y_i$, $1 \leq i \leq n$. Define a multiplicative map from F_{n+1} to \mathcal{B} as follows.

$$\tau : F_{n+1} \longrightarrow \mathcal{B}$$

$$(2.5) \quad \tau(w) = \begin{cases} \frac{1}{\Gamma}y_i - 1 & w = e_i, 1 \leq i \leq n \\ \frac{1}{\Gamma\mu}y_i - 1 & w = e_i^{-1}, 1 \leq i \leq n \\ e^{\pm 1} & w = e^{\pm 1} \\ 1 & w = 1 \end{cases}$$

Clearly $\tau(e_i)\tau(e_i^{-1}) = 1 = \tau(1)$ in \mathcal{B} . Therefore, we can extend the action of τ uniquely to arbitrary words to get a well-defined multiplicative map on F_{n+1} . Actually τ extends to an algebra morphism from the group ring $R[F_{n+1}]$ to \mathcal{B} .

Next, for $1 \leq i, j \leq n$, we define an R -linear map $\alpha_{ij} : R\langle e^{\pm 1}, y_1, y_2, \dots, y_n \rangle \longrightarrow \mathcal{A}_n$,

$$\alpha_{ij}(e^{i_1}y_{j_1}e^{i_2}y_{j_2}\cdots e^{i_k}y_{j_k}e^{i_{k+1}}) := a_{i,j_1}^{i_1}a_{j_1,j_2}^{i_2}\cdots a_{j_{k-1},j_k}^{i_k}a_{j_k,j}^{i_{k+1}}$$

It's easy to check that α_{ij} factors through \mathcal{I} because of the fact that $a_{ii}^0 = (1 + \mu)\Gamma$. Therefore we get an induced map from \mathcal{B} to \mathcal{A}_n , which is still denoted by α_{ij} .

Finally we can describe the isomorphism between $\tilde{\mathcal{A}}_n$ and \mathcal{A}_n .

Let δ_i be the straight line from z_0 to q_i , and $\bar{\delta}_i$ be the same line but with reversed direction. For any curve $\gamma \in \mathcal{Q}_n$ with $\gamma(0) = q_i, \gamma(1) = q_j$, let $\tilde{\gamma} = \delta_i * \gamma * \bar{\delta}_j$, then $\tilde{\gamma}$ becomes an element in $\pi_1(D_n, z_0) = F_{n+1}$. Define the isomorphism $\psi : \tilde{\mathcal{A}}_n \longrightarrow \mathcal{A}_n$ by $\psi(\gamma) := \alpha_{ij}\tau(\tilde{\gamma})$.

Theorem 2. The map ψ defined above is an algebra isomorphism from $\tilde{\mathcal{A}}_n$ to \mathcal{A}_n sending γ_{ij}^x to a_{ij}^x .

Proof Clearly, $\psi(\gamma)$ is independent of the choice of γ in the equivalence class.

We first show ψ factors through the “skein” relations.

It's easy to see that $\psi(\gamma_{ij}^x) = a_{ij}^x$. In particular, $\psi(\gamma_{ii}^0) = a_{ii}^0 = (1 + \mu)\Gamma$, so the first “skein” relation is passed to an identity under ψ .

Let C_1, C_2 denote the two curves passing above and below p_k , respectively, in the definition of the second “skein” relation in Figure 5. They have the same initial and end points, say q_i, q_j . Let C_3, C_4 be the curves which ends at q_k and starts at q_k , respectively. So C_3 starts from q_i and C_4 ends at q_j . Let w_3, w_4 be the words in F_{n+1} which represent \tilde{C}_3, \tilde{C}_4 , then it's clear that the words which represent \tilde{C}_1, \tilde{C}_2 are $w_3w_4, w_3e_kw_4$.

Therefore, $\psi(C_1) + \psi(C_2) = \alpha_{ij}(\tau(w_3)\tau(w_4)) + \alpha_{ij}(\tau(w_3)(\frac{1}{\Gamma}y_k - 1)\tau(w_4)) = \frac{1}{\Gamma}\alpha_{ij}(\tau(w_3)y_k\tau(w_4)) = \frac{1}{\Gamma}\alpha_{ik}(\tau(w_3))\alpha_{kj}(\tau(w_4)) = \frac{1}{\Gamma}\psi(C_3)\psi(C_4)$, which says ψ factors through the second “skein” relation.

The above arguments show that ψ is a well-defined algebra morphism. Define the inverse map $\psi' : \mathcal{A}_n \longrightarrow \tilde{\mathcal{A}}_n$ by sending each a_{ij}^x to γ_{ij}^x . Noting that γ_{ij}^x are generators of $\tilde{\mathcal{A}}_n$, it's obvious that $\psi\psi' = Id$ and $\psi'\psi = Id$. Therefore, ψ is an algebra isomorphism. \square

Now we describe a natural action of \mathcal{C}_n on $\tilde{\mathcal{A}}_n$.

Recall that the group of isotopy classes of homeomorphisms of D_n with boundary fixed point-wise is the classical braid group on $n + 1$ strands \mathcal{B}_{n+1} .² Here we assume the generators are $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$, where σ_0 is the Dehn twist that switches p with p_1 counter clock-wise and σ_i switches p_i with p_{i+1} , $1 \leq i \leq n-1$. Also recall that we identified \mathcal{C}_n with the subgroup of \mathcal{B}_{n+1} which consists of the braids that fix the first puncture. See Section 2.1 for the explicit embedding. Therefore, the elements of \mathcal{C}_n fix the puncture p and permute $\{p_i, 1 \leq i \leq n\}$. We can furthermore stipulate that the horizontal line segments $p_i q_i$ remain horizontal and of fixed length during the isotopy, so that the elements of \mathcal{C}_n also permute the q_i 's. It follows that the elements of \mathcal{C}_n act on \mathcal{Q}_n . It's also easy to see that this action actually preserves the “skein” relations. Therefore, we get a natural action $\tilde{\Phi}$ of \mathcal{C}_n on $\tilde{\mathcal{A}}_n$.

Theorem 3. The algebra isomorphism $\psi : \tilde{\mathcal{A}}_n \rightarrow \mathcal{A}_n$ preserves the action of \mathcal{C}_n , i.e. $\psi \tilde{\Phi}_\beta = \Phi_\beta \psi$, for any $\beta \in \mathcal{C}_n$.

Proof It suffices to check for any $\beta = \alpha_k$, $\psi \tilde{\Phi}_\beta = \Phi_\beta \psi$ holds on the generators γ_{ij}^x . We left this as an exercise. \square

Remark 4. It's worth pointing out that when we want to find the image of some complicated curve in $\tilde{\mathcal{A}}_n$ under ψ , it's usually more efficient to use the “skein” relations than using the definition directly. Also, instead of memorizing the action of \mathcal{C}_n on the a_{ij}^x 's, it's much easier to manipulate the “skein” relations and the Dehn twists. This provides us another way to calculate the action of a braid β on a_{ij}^x , namely, first use a sequence of Dehn twists representing β to map γ_{ij}^x to some curve, and then decompose this curve into a polynomial of generators using “skein” relations, finally replace the generators in the polynomial by the corresponding a_{ij}^x 's.

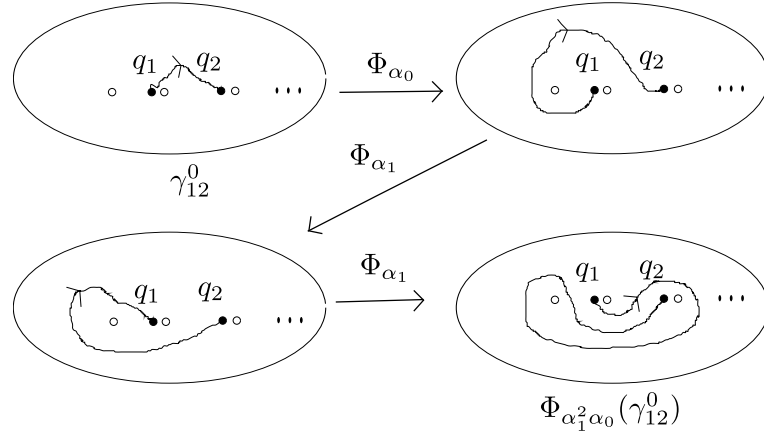
For example, to obtain $\Phi_{\alpha_1^2 \alpha_0}(a_{12}^0)$, we first compute $\tilde{\Phi}_{\alpha_1^2 \alpha_0}(\gamma_{12}^0)$ using Dehn twists that represent $\alpha_1^2 \alpha_0$. See Figure 9. Then we decompose the resulting curve using “skein” relations to get the expression.

$$\begin{aligned} \tilde{\Phi}_{\alpha_1^2 \alpha_0}(\gamma_{12}^0) &= \gamma_{12}^{-1} - \frac{1}{\Gamma} \gamma_{11}^{-1} \gamma_{12}^0 + \frac{1}{\Gamma} \gamma_{12}^0 \gamma_{22}^{-1} - \frac{1}{\Gamma^2 \mu} \gamma_{12}^0 \gamma_{21}^0 \gamma_{12}^{-1} - \frac{1}{\Gamma^2} \gamma_{12}^0 \gamma_{21}^{-1} \gamma_{12}^0 \\ &+ \frac{1}{\Gamma^3 \mu} \gamma_{12}^0 \gamma_{21}^0 \gamma_{11}^{-1} \gamma_{12}^0 \end{aligned}$$

Replacing the γ_{ij}^x 's above with a_{ij}^x 's, we obtain the expression for $\Phi_{\alpha_1^2 \alpha_0}(a_{12}^0)$.

There are analogous topological interpretations of the extended actions of \mathcal{C}_n on \mathcal{A}_n^+ and \mathcal{A}_n^- .

²Note that here D_n has $n + 1$ punctures.


 FIGURE 9. $\Phi_{\alpha_1^2 \alpha_0}(\gamma_{12}^0)$

The procedure goes the same as above, and we will only point out what modifications should be made at each step.

First of all, let D_n^+ be the the punctured disk with punctures p, p_0, p_1, \dots, p_n arranged from left to right and similarly let D_n^- be the punctured disk with punctures $p, p_1, \dots, p_n, p_{n+1}$. Also in both cases, still choose the points $q_i = p_i - \epsilon$, for some tiny $\epsilon > 0$. Let \mathcal{Q}_n^\pm be the set of equivalence classes of curves in D_n^\pm which start and end at the q_i 's. Define $\tilde{\mathcal{A}}_n^\pm$ to be the R -algebra generated by elements of \mathcal{Q}_n^\pm modulo the “skein” relations in Figure 10:

$$\begin{aligned}
 1). \quad & \text{Diagram of a loop around } q_i = (1 + \mu)\Gamma \quad 1 \leq i \leq n \text{ or } q_i = q_\pm \\
 2). \quad & \text{Diagram of a curve passing through } q_i + \text{Diagram of a curve passing through } q_i = \frac{1}{\Gamma} \text{Diagram of a curve passing through } q_i \otimes \text{Diagram of a curve passing through } q_i \quad 1 \leq i \leq n \\
 3). \quad & \text{Diagram of a curve passing through } q_\pm = \text{Diagram of a curve passing through } q_\pm
 \end{aligned}$$

FIGURE 10. skein relation

where $q_\pm = q_0$ in the “+” case and $q_\pm = q_{n+1}$ otherwise.

So we added one more relation when defining $\tilde{\mathcal{A}}_n^\pm$, namely, the curves are allowed to pass through the new puncture $p_0(p_{n+1})$.

The fundamental group of D_n^\pm is the free group F_{n+2} generated by $e, e', e_i, 1 \leq i \leq n$, where e' is the generator that correspond to the new

puncture p_0 or p_{n+1} . We will use the same intermediate algebra \mathcal{B} , and the map τ is extended to F_{n+2} by furthermore defining $\tau(e') = 1$.

In the same way as we defined the isomorphism ψ from $\tilde{\mathcal{A}}_n$ to \mathcal{A}_n , we can define an isomorphism ψ^\pm from $\tilde{\mathcal{A}}_n^\pm$ to \mathcal{A}_n^\pm which sends γ_{ij}^x to a_{ij}^x .

Next, we extend the action of \mathcal{C}_n to $\tilde{\mathcal{A}}_n^\pm$.

Recall the embedding $\epsilon^+ : \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$ introduced in Section 2.1. For notational convenience, we denote the generators of \mathcal{C}_{n+1} by $\alpha_{-1}, \alpha_0, \dots, \alpha_{n-1}$. Thus the embedding ϵ^+ sends α_0 to $\alpha_0 \alpha_{-1} \alpha_0$ and α_i to α_i , $1 \leq i \leq n-1$. From the geometrical point of view, ϵ^+ simply inserts a strand labeled by p_0 right next to $\{p\} \times [0, 1]$. See the first picture in Figure 11.

It's easy to see any braid in $\epsilon^+(\mathcal{C}_n)$ fixes the first two punctures (the punctures that are labeled by p and p_0). Thus it should be clear that via the embedding ϵ^+ , the action of \mathcal{C}_n preserves all the “skein” relations defining $\tilde{\mathcal{A}}_n^+$, and therefore induces an action $\tilde{\Phi}^+$ on $\tilde{\mathcal{A}}_n^+$.

For the action $\tilde{\Phi}^-$ of \mathcal{C}_n on $\tilde{\mathcal{A}}_n^-$, we use the other embedding $\epsilon^- : \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$. Note that here the generators of \mathcal{C}_{n+1} are $\alpha_0, \dots, \alpha_n$, and $\epsilon^-(\alpha_i) = \alpha_i$, $0 \leq i \leq n-1$. And the map ϵ^- inserts a strand labeled by p_{n+1} on the right of the braid. See the second picture in Figure 11.

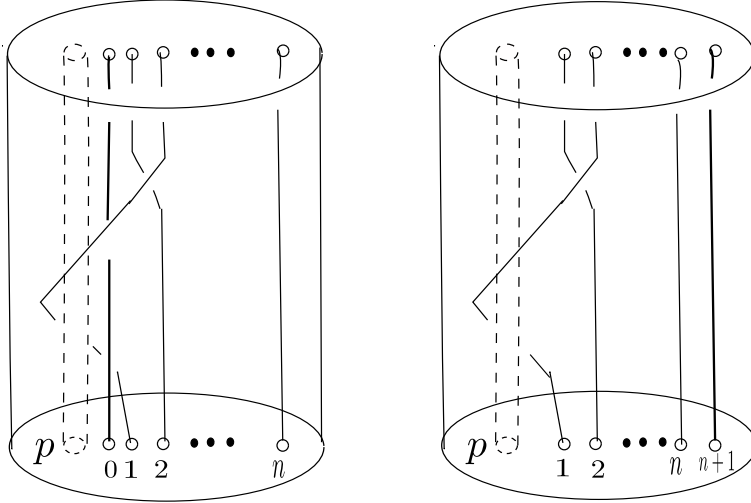


FIGURE 11. $\epsilon^+(\alpha_1 \alpha_0)$ and $\epsilon^-(\alpha_1 \alpha_0)$

Here in Figure 11 we use i to represent p_i .

Again, since elements of $\epsilon^-(\mathcal{C}_n)$ fix p_{n+1} , they preserve the “skein” relations that define $\tilde{\mathcal{A}}_n^-$. We thus get an induced action $\tilde{\Phi}^-$ of \mathcal{C}_n on $\tilde{\mathcal{A}}_n^-$.

$\tilde{\mathcal{A}}_n$ can obviously be embedded as a subalgebra into $\tilde{\mathcal{A}}_n^\pm$. We have the following theorem which relates the topological interpretations of the actions of \mathcal{C}_n to the algebraic interpretations.

Theorem 4. The maps $\psi^\pm : \tilde{\mathcal{A}}_n^\pm \longrightarrow \mathcal{A}_n^\pm$ are algebra isomorphisms and commute with the extended actions of \mathcal{C}_n , namely, for any $\beta \in \mathcal{C}_n$, $\psi^\pm \tilde{\Phi}_\beta^\pm = \Phi_\beta^\pm \psi^\pm$. Moreover, $(\tilde{\Phi}_\beta^\pm)|_{\tilde{\mathcal{A}}_n} = \tilde{\Phi}_\beta$, and the following diagram commutes:

$$(2.6) \quad \begin{array}{ccc} \tilde{\mathcal{A}}_n & \xrightarrow{\psi} & \mathcal{A}_n \\ \downarrow & & \downarrow \\ \tilde{\mathcal{A}}_n^\pm & \xrightarrow{\psi^\pm} & \mathcal{A}_n^\pm \end{array}$$

And each of the maps in the above diagram preserves the action of \mathcal{C}_n .

Proof Proofs are analogous to that of Theorem 2. \square

It is worth noting that the actions of $\tilde{\Phi}^\pm$ on $\tilde{\mathcal{A}}_n^\pm$ and the action of $\tilde{\Phi}$ on $\tilde{\mathcal{A}}_n$ can also be visualized as follows.

For a braid $\beta \in \mathcal{C}_n$, draw a braid diagram of β inside $D_n \times [0, 1]$, such that the intersection of the braid with $D_n \times \{0, 1\}$ are exactly the punctures p_i 's. Perturb the braid diagram to get a parallel copy of it such that the intersection of the copy with $D_n \times \{0, 1\}$ are the q_i 's. For any curve $\gamma \subset D_n \times \{0\}$ representing some element in $\tilde{\mathcal{A}}_n$, slide γ along the copy diagram in the complement of the braid diagram until it reaches $D_n \times \{1\}$, then the resulting curve is $\tilde{\Phi}_\beta(\gamma)$.

To visualize $\tilde{\Phi}_\beta^\pm$, we draw a braid diagram of $\epsilon^\pm(\beta)$ inside $D_n^\pm \times I$, make a parallel copy of it, and slide any curve along the copy diagram up to $D_n^\pm \times \{1\}$.

With the above observations, we have the following simple but important proposition.

Proposition 1. Let $\beta \in \mathcal{C}_n$ be a braid.

- 1). For any two curves $\gamma_1, \gamma_2 \in \mathcal{Q}_n^+$, such that $\gamma_1(1) = \gamma_2(0) = q_0$ and $\gamma_1(0) = q_i, \gamma_2(1) = q_j$ for some $1 \leq i, j \leq n$, then $\gamma_1 * \gamma_2$ is a curve in \mathcal{Q}_n from q_i to q_j , and we have $\tilde{\Phi}_\beta(\gamma_1 * \gamma_2) = \tilde{\Phi}_\beta^+(\gamma_1) * \tilde{\Phi}_\beta^+(\gamma_2)$.
- 2). For any two curves $\gamma_1, \gamma_2 \in \mathcal{Q}_n^-$, such that $\gamma_1(1) = \gamma_2(0) = q_{n+1}$ and $\gamma_1(0) = q_i, \gamma_2(1) = q_j$ for some $1 \leq i, j \leq n$, then $\gamma_1 * \gamma_2$ is a curve in \mathcal{Q}_n from q_i to q_j , and we have $\tilde{\Phi}_\beta(\gamma_1 * \gamma_2) = \tilde{\Phi}_\beta^-(\gamma_1) * \tilde{\Phi}_\beta^-(\gamma_2)$.

Here, and throughout the paper, $*$ again means connecting the two curves.

Remark 5. From now on, we will identify $\tilde{\mathcal{A}}_n$ with \mathcal{A}_n , $\tilde{\mathcal{A}}_n^\pm$ with \mathcal{A}_n^\pm , γ_{ij}^x with a_{ij}^x via the corresponding isomorphisms and identify $\tilde{\Phi}_\beta$ with Φ_β , $\tilde{\Phi}_\beta^\pm$ with Φ_β^\pm , respectively. A useful picture to keep in mind is as follows. a_{ij}^x is the left arc diagram described in Figure 6. The action Φ_β (Φ_β^\pm) of β on some curve is to slide that curve along the parallel copy of the braid diagram that represents β ($\epsilon^\pm(\beta)$) up to $D_n \times \{1\}$ ($D_n^\pm \times \{1\}$).

We can also define the “ $*$ ” operation on some elements of \mathcal{A}_n .

Definition 1. 1). Let $P, Q \in \mathcal{A}_n^+$ such that $P = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n P_i^x a_{i0}^x$, $Q = \sum_{y \in \mathbb{Z}} \sum_{j=1}^n a_{0j}^y Q_j^y$, $P_i^x, Q_j^y \in \mathcal{A}_n$, then $P * Q \in \mathcal{A}_n$ is defined to be $\sum_{x, y \in \mathbb{Z}} \sum_{i, j=1}^n P_i^x a_{ij}^{x+y} Q_j^y$.

2). Let $P, Q \in \mathcal{A}_n^-$ such that $P = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n P_i^x a_{i, n+1}^x$, $Q = \sum_{y \in \mathbb{Z}} \sum_{j=1}^n a_{n+1, j}^y Q_j^y$, $P_i^x, Q_j^y \in \mathcal{A}_n$, then $P * Q \in \mathcal{A}_n$ is defined to be $\sum_{x, y \in \mathbb{Z}} \sum_{i, j=1}^n P_i^x a_{ij}^{x+y} Q_j^y$.

3). Two elements $P, Q \in \mathcal{A}_n^\pm$ are called connectable, if they satisfy the condition in one of the above two definitions.

Proposition 2. If $P, Q \in \mathcal{A}_n^\pm$ are connectable, then for any $\beta \in \mathcal{C}_n$, $\Phi_\beta^\pm(P), \Phi_\beta^\pm(Q)$ are also connectable, and $\Phi_\beta(P * Q) = \Phi_\beta^\pm(P) * \Phi_\beta^\pm(Q)$.

Proof We will only prove the “ $+$ ” case. The proof of the other case is analogous.

Let P, Q be as described in 1) of Definition 1, then for $\beta \in \mathcal{C}_n$, $\Phi_\beta^+(P) = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n \Phi_\beta^+(P_i^x) \Phi_\beta^+(a_{i0}^x) = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n \Phi_\beta(P_i^x) \Phi_\beta^+(a_{i0}^x)$, and similarly, $\Phi_\beta^+(Q) = \sum_{y \in \mathbb{Z}} \sum_{j=1}^n \Phi_\beta^+(a_{0j}^y) \Phi_\beta(Q_j^y)$. Clearly, $\Phi_\beta^+(a_{i0}^x)$ and $\Phi_\beta^+(a_{0j}^y)$ are connectable, so $\Phi_\beta^+(P)$ and $\Phi_\beta^+(Q)$ are connectable. Moreover,

$$\begin{aligned} \Phi_\beta^+(P) * \Phi_\beta^+(Q) &= \sum_{x, y \in \mathbb{Z}} \sum_{i, j=1}^n \Phi_\beta(P_i^x) \{ \Phi_\beta^+(a_{i0}^x) * \Phi_\beta^+(a_{0j}^y) \} \Phi_\beta(Q_j^y) \\ &\stackrel{\text{Proposition 1}}{=} \sum_{x, y \in \mathbb{Z}} \sum_{i, j=1}^n \Phi_\beta(P_i^x) \Phi_\beta(a_{ij}^{x+y}) \Phi_\beta(Q_j^y) = \\ &\Phi_\beta\left(\sum_{x, y \in \mathbb{Z}} \sum_{i, j=1}^n P_i^x a_{ij}^{x+y} Q_j^y\right) = \Phi_\beta(P * Q). \end{aligned}$$

□

3. THE FRAMED KNOT INVARIANT

From now on, we will assume the closure of $\beta \in \mathcal{C}_n$ is a knot in $S^1 \times S^2$.

In this section, first we give the definition of the framed knot invariant. Since the knot invariant looks very complicated at first glance, we will compute some examples after the definition. We then proceed to give some ancillary results, and finally prove the invariance under Markov moves.

3.1. Definition of the invariant. Here are some notations we will use to define the invariant.

Let $M_\infty(\mathcal{A}_n)$ denote the set of $\infty \times \infty$ matrices with elements in \mathcal{A}_n , namely, the rows and columns of a matrix in $M_\infty(\mathcal{A}_n)$ are both indexed by integers. We call a matrix *row-finite* if there are only finitely many non-zero entries in each row. A *column-finite* matrix is defined analogously. If M, N are two matrices in $M_\infty(\mathcal{A}_n)$, in general the multiplication of them is not well-defined. However, if M is *row-finite* or N is *column-finite*, then $M \cdot N$ is well-defined. And the associativity is satisfied whenever multiplications make sense. All throughout the paper, the matrices always satisfy the above condition when they are multiplied together, and for $x, y \in \mathbb{Z}$, we will use M^{xy} to refer to the (x, y) -entry of M . We will also use an element $c \in \mathcal{A}_n$ to represent the scalar matrix in $M_\infty(\mathcal{A}_n)$ which has entry c on the diagonal and 0 elsewhere. Let $M_n(M_\infty(\mathcal{A}_n))$ denote the set of $n \times n$ matrices with entries in $M_\infty(\mathcal{A}_n)$.

Recall $\epsilon^\pm : \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$ are the two embeddings, and for $\beta \in \mathcal{C}_n$, $(\Phi_\beta^-)|_{\mathcal{A}_n} = \Phi_\beta = (\Phi_\beta^+)|_{\mathcal{A}_n}$.

It's not hard to see (perhaps easier from the topological interpretation) that for $1 \leq i \leq n, x \in \mathbb{Z}$, $\Phi_\beta^-(a_{i,n+1}^x)$ can be written as a finite linear combinations of $a_{k,n+1}^z, 1 \leq k \leq n, z \in \mathbb{Z}$ with coefficients in \mathcal{A}_n . A similar argument holds for $\Phi_\beta^-(a_{n+1,i}^x), \Phi_\beta^+(a_{i,0}^x), \Phi_\beta^+(a_{0,i}^x)$. For example, $\Phi_\beta^+(a_{0,i}^x)$ is a finite linear combinations of $a_{0,k}^z$ with coefficients in \mathcal{A}_n multiplied on the right. Explicitly, this is how we define $\Phi_\beta^{-L}, \Phi_\beta^{-R}, \Phi_\beta^{+L}, \Phi_\beta^{+R} \in M_n(M_\infty(\mathcal{A}_n))$.

For each $\beta \in \mathcal{C}_n, 1 \leq i, j \leq n, x, y \in \mathbb{Z}$, define

$$\begin{aligned} \Phi_\beta^-(a_{i,n+1}^x) &= \sum_{k=1}^n \sum_{z \in \mathbb{Z}} (\Phi_\beta^{-L})_{ik}^{xz} a_{k,n+1}^z \\ \Phi_\beta^-(a_{n+1,j}^y) &= \sum_{k=1}^n \sum_{z \in \mathbb{Z}} a_{n+1,k}^z (\Phi_\beta^{-R})_{kj}^{zy} \\ \Phi_\beta^+(a_{i,0}^x) &= \sum_{k=1}^n \sum_{z \in \mathbb{Z}} (\Phi_\beta^{+L})_{ik}^{xz} a_{k,0}^z \end{aligned}$$

$$\Phi_{\beta}^{+}(a_{0,j}^y) = \sum_{k=1}^n \sum_{z \in \mathbb{Z}} a_{0,k}^z (\Phi_{\beta}^{+R})_{kj}^{zy}$$

where $(\Phi_{\beta}^{-L})_{ik}^{xz}$ is the (x, z) -entry of the $\infty \times \infty$ matrix $(\Phi_{\beta}^{-L})_{ik}$ which is the (i, k) -entry of the $n \times n$ matrix Φ_{β}^{-L} . So we have $\Phi_{\beta}^{-L} \in M_n(M_{\infty}(\mathcal{A}_n))$. A similar statement holds for the other three symbols.

Define $a_{ij} \in M_{\infty}(\mathcal{A}_n)$ by $(a_{ij})^{xy} = a_{ij}^{x+y}$ and define $A \in M_n(M_{\infty}(\mathcal{A}_n))$ by $A_{ij} = a_{ij}$.

Lemma 2. For $\beta \in \mathcal{C}_n, 1 \leq i, j \leq n$, $(\Phi_{\beta}^{-L})_{ij}, (\Phi_{\beta}^{+L})_{ij}$ are *row-finite* and $(\Phi_{\beta}^{-R})_{ij}, (\Phi_{\beta}^{+R})_{ij}$ are *column-finite*.

Proof These are direct consequences of the definitions. \square

Remark 6. Actually, $(\Phi_{\beta}^{+L})_{ij}, (\Phi_{\beta}^{+R})_{ij}$ are both *row-finite* and *column-finite*. This is due to a careful inspection of the action Φ_{β}^{+} . We will not use this property though.

For $1 \leq p, q \leq n, f \in \mathbb{Z}$, let $\Lambda_{f;p,q} \in M_n(M_{\infty}(\mathcal{A}_n))$ be the diagonal matrix with the (p, p) -th entry λ , the (q, q) -entry μ^{-f} and other diagonal entries 1.

Definition 2. Let $\beta \in \mathcal{C}_n, 1 \leq p, q \leq n, f \in \mathbb{Z}$, then $HC_0(\beta; f; p, q)$ is defined to be the R -algebra \mathcal{A}_n modulo the two sided ideal $\mathcal{I}_{\beta;f;p,q}$ generated by the entries of the entries of following matrices:

$$\begin{aligned} A - \Lambda_{f;p,q} \Phi_{\beta}^{-L} A \\ A - A \Phi_{\beta}^{-R} \Lambda_{f;p,q}^{-1} \\ A - \Lambda_{f;p,q} \Phi_{\beta}^{+L} A \\ A - A \Phi_{\beta}^{+R} \Lambda_{f;p,q}^{-1} \end{aligned}$$

Remark 7. (1) For a matrix $M \in M_n(M_{\infty}(\mathcal{A}_n))$, the phrase “the entries of the entries of M ” is really awkward. From now on, we will use “the elements of M ” to stand for “the entries of the entries of M ”.

$$\begin{aligned} (2) \text{ Note that } (\Lambda_{p,q;f} \Phi_{\beta}^{-L} A)_{ij}^{xy} &= \sum_{k=1}^n \sum_{z \in \mathbb{Z}} \lambda^{\delta_{i,p}} \mu^{-f \delta_{i,q}} (\Phi_{\beta}^{-L})_{ik}^{xz} A_{kj}^{zy} \\ &= \sum_{k=1}^n \sum_{z \in \mathbb{Z}} \lambda^{\delta_{i,p}} \mu^{-f \delta_{i,q}} ((\Phi_{\beta}^{-L})_{ik}^{xz} a_{k,n+1}^z) * a_{n+1,j}^y \\ &= \lambda^{\delta_{i,p}} \mu^{-f \delta_{i,q}} \Phi_{\beta}^{-}(a_{i,n+1}^x) * a_{n+1,j}^y. \end{aligned}$$

Since $A_{ij}^{xy} = a_{ij}^{x+y} = a_{i,n+1}^x * a_{n+1,j}^y$, the relations in $\mathcal{I}_{\beta;f;p,q}$ are the same as the following:

$$a_{i,n+1}^x * a_{n+1,j}^y - \lambda^{\delta_{i,p}} \mu^{-f \delta_{i,q}} \Phi_{\beta}^{-}(a_{i,n+1}^x) * a_{n+1,j}^y,$$

$$\begin{aligned} & a_{i,n+1}^x * a_{n+1,j}^y - \lambda^{-\delta_{j,p}} \mu^{f\delta_{j,q}} a_{i,n+1}^x * \Phi_{\beta}^{-}(a_{n+1,j}^y), \\ & a_{i,0}^x * a_{0,j}^y - \lambda^{\delta_{i,p}} \mu^{-f\delta_{i,q}} \Phi_{\beta}^{+}(a_{i,0}^x) * a_{0,j}^y, \\ & a_{i,0}^x * a_{0,j}^y - \lambda^{-\delta_{j,p}} \mu^{f\delta_{j,q}} a_{i,0}^x * \Phi_{\beta}^{+}(a_{0,j}^y), \forall 1 \leq i, j \leq n, x, y \in \mathbb{Z}. \end{aligned}$$

Note that for $\beta \in C_n$, it has a natural action by permutation on the set $\{1, \dots, n\}$. Our convention here is that the braid diagram always goes upward, and if the i -th strand ends at the j -th position, then $\beta(i) = j$.

Lemma 3. For $\beta \in C_n, 1 \leq p, q \leq n, f \in \mathbb{Z}$, we have $HC_0(\beta; f; p, q) \simeq HC_0(\beta; f; \beta(p), q) \simeq HC_0(\beta; f; p, \beta(q))$.

Proof Define $\psi : HC_0(\beta; f; p, q) \longrightarrow HC_0(\beta; f; \beta(p), q)$ by $\psi(a_{ij}^x) = \lambda^{-\delta_{i,\beta(p)}} a_{ij}^x \lambda^{\delta_{j,\beta(p)}}$. We need to check that ψ sends $\mathcal{I}_{\beta;f;p,q}$ to $\mathcal{I}_{\beta;f;\beta(p),q}$.

Note that $\Phi_{\beta}^{-}(a_{i,n+1}^x) * a_{n+1,j}^y$ can be written as a non-commutative polynomial in which each monomial is of the form $a_{\beta(i),i_1}^{x_1} a_{i_1,i_2}^{x_2} \cdots a_{i_k,j}^{x_{k+1}}$, thus we have

$$\begin{aligned} & \psi(a_{ij}^{xy} - \lambda^{\delta_{i,p}} \mu^{-f\delta_{i,q}} \Phi_{\beta}^{-}(a_{i,n+1}^x) * a_{n+1,j}^y) \\ &= \lambda^{-\delta_{i,\beta(p)}} a_{ij}^{xy} \lambda^{\delta_{j,\beta(p)}} - \lambda^{\delta_{i,p}} \mu^{-f\delta_{i,q}} \lambda^{-\delta_{\beta(i),\beta(p)}} \Phi_{\beta}^{-}(a_{i,n+1}^x) * a_{n+1,j}^y \lambda^{\delta_{j,\beta(p)}} \\ &= \lambda^{-\delta_{i,\beta(p)}} (a_{ij}^{xy} - \lambda^{\delta_{i,\beta(p)}} \mu^{-f\delta_{i,q}} \Phi_{\beta}^{-}(a_{i,n+1}^x) * a_{n+1,j}^y) \lambda^{\delta_{j,\beta(p)}} \in \mathcal{I}_{\beta;f;\beta(p),q}. \end{aligned}$$

The other three relations in $\mathcal{I}_{\beta;f;p,q}$ can be shown analogously that they are mapped under ψ to $\mathcal{I}_{\beta;f;\beta(p),q}$. Thus the map ψ is well defined. It's also clear that it's an isomorphism.

The isomorphism $HC_0(\beta; f; p, q) \simeq HC_0(\beta; f; p, \beta(q))$ can be proved similarly by sending a_{ij}^x to $\mu^{k\delta_{i,\beta(p)}} a_{ij}^x \mu^{-k\delta_{j,\beta(p)}}$. \square

Corollary 1. If the closure of $\beta \in C_n$ is a knot in $S^1 \times S^2$, then $HC_0(\beta; f; p, q)$ is independent of the values of p, q .

Apparently from the definition, $HC_0(\beta; f; p, p)$ can be obtained from $HC_0(\beta; 0; p, p)$ by replacing λ by $\lambda\mu^{-f}$. We will use the notations $HC_0(\beta; f; p) = HC_0(\beta; f; p, p)$, $HC_0(\beta; f) = HC_0(\beta; f; 1, 1)$ and $HC_0(\beta) = HC_0(\beta; 0; 1, 1)$. By the corollary above, $HC_0(\beta; f; p)$ is dependent of the choice of p , so we have $HC_0(\beta; f) \simeq HC_0(\beta; f; p)$ for any p .

The following theorem is our main result.

Theorem 5. Let $\beta, \alpha \in C_n, f \in \mathbb{Z}$ such that the closure of β in $S^1 \times S^2$ is a knot, then we have the following isomorphisms:

- 1). $HC_0(\beta; f) \simeq HC_0(\alpha^{-1}\beta\alpha; f)$;
- 2). $HC_0(\beta; f) \simeq HC_0(\epsilon^{-}(\beta)\alpha_n; f-1) \simeq HC_0(\epsilon^{-}(\beta)\alpha_n^{-1}; f+1)$;
- 3). $HC_0(\beta; f) \simeq HC_0(\epsilon^{+}(\beta)\alpha_n; f-1) \simeq HC_0(\epsilon^{+}(\beta)\alpha_n^{-1}; f+1)$;

We will give a proof of the theorem in Section 3.4.

Endow $S^1 \times S^2$ with the standard orientation. Let K be a framed oriented knot in $S^1 \times S^2$ with l, m the homotopy classes of the longitude

and the meridian of K in $\pi_1(S^1 \times S^2 \setminus K)$. The orientations of K and $S^1 \times S^2$ determine the meridian class m uniquely. More precisely, let $\nu(K)$ be the tubular neighborhood of K , which is homeomorphic to $K \times D^2$. Choose an orientation on D^2 so that the homeomorphism of $\nu(K)$ with $K \times D^2$ is orientation preserving. Then for any $z \in K$, the image of $z \times \partial D^2$ under the homeomorphism determines the meridian class. Assume K is represented by the closure of a braid $\beta \in \mathcal{C}_n$, and that $[l] = [\hat{\beta}'][m]^f$, where β' is a parallel push-off copy of β , then $HC_0(K; l)$ is defined to be $HC_0(\beta; f)$.

Corollary 2. $HC_0(K; l)$ as an R -algebra is a framed knot invariant for knots in $S^1 \times S^2$.

Proof For a braid diagram $\beta \in \mathcal{C}_n$, let β' be the parallel push-off copy of β . Then we have $[\hat{\beta}'][m]^{\pm 1} = [(\epsilon^+(\beta)\alpha_n^{\pm 1})']$, $[\hat{\beta}'][m]^{\pm 1} = [(\epsilon^-(\beta)\alpha_n^{\pm 1})']$ and for any $\alpha \in \mathcal{C}_n$, we have $[\hat{\beta}'] = [(\alpha^{-1}\beta\alpha)']$. \square

Remark 8. The invariant $HC_0(K; l)$ is conjectured to be the 0-th knot contact homology of K , which is defined to be the 0-th Legendrian contact homology of Λ_K in $ST^*(S^1 \times S^2)$, where $ST^*(S^1 \times S^2)$ is the unit cotangent bundle of $S^1 \times S^2$ and Λ_K is the unit conormal bundle of K . As this paper is not relevant to proving this conjecture, readers should just treat HC_0 purely as a name.

3.2. Examples. Before proving invariance, we first look at some examples.

Example 1. 1). **Unknot.** The most simple example is the unknot represented by the identity element e in \mathcal{C}_1 . We compute $HC_0(e; f)$ for $f \in \mathbb{Z}$. In this case, it's clear that $\Phi_e^{+L}, \Phi_e^{+R}, \Phi_e^{-L}, \Phi_e^{-R}$ are all identity matrices, thus all the relations in $\mathcal{I}_{e;f;1,1}$ become $(1 - \lambda\mu^{-f})a_{11}^x$, and so $HC_0(e; f) \simeq R\langle a_{11}^x, x \in \mathbb{Z} \rangle / \langle (1 - \lambda\mu^{-f})a_{11}^x \rangle$.

2). **α_0^2 .** Set $\beta = \alpha_0^2, \Lambda = \Lambda_{\beta;0;1,1}$. We first compute $\Phi_\beta^{+L}, \Phi_\beta^{+R}$. Direct calculations show that $\Phi_\beta^+(a_{10}^x) = \mu^2 a_{10}^{x-2}, \Phi_\beta^+(a_{01}^y) = \mu^{-2} a_{01}^{y+2}$. Thus we have $(\Phi_\beta^{+L})_{11}^{xy} = \mu^2 \delta_{x-2,y}, (\Phi_\beta^{+R})_{11}^{xy} = \mu^{-2} \delta_{x-2,y}$, and therefore $(\Lambda \Phi_\beta^{+L} A)_{11}^{xy} = \lambda \mu^2 a_{11}^{x+y-2}, (A \Phi_\beta^{+R} \Lambda^{-1})_{11}^{xy} = (\lambda \mu^2)^{-1} a_{11}^{x+y+2}$. So the third and fourth relation defining $\mathcal{I}_{\beta;0;1,1}$ both are $a_{11}^{x+2} - \lambda \mu^2 a_{11}^x, x \in \mathbb{Z}$.

Now we compute $\Phi_\beta^{-L}, \Phi_\beta^{-R}$. By definition, $\Phi_{\alpha_0}^-(a_{11}^x) = a_{11}^x, \Phi_{\alpha_0}^-(a_{12}^x) = -\mu a_{12}^{x-1} + \frac{1}{\Gamma} a_{11}^x a_{12}^{-1}$. Therefore,

$$\begin{aligned} \Phi_{\alpha_0^2}^-(a_{12}^x) &= -\mu \Phi_{\alpha_0}^-(a_{12}^{x-1}) + \frac{1}{\Gamma} \Phi_{\alpha_0}^-(a_{11}^x) \Phi_{\alpha_0}^-(a_{12}^{-1}) \\ &= \mu^2 a_{12}^{x-2} - \frac{\mu}{\Gamma} a_{11}^{x-1} a_{12}^{-1} - \frac{\mu}{\Gamma} a_{11}^x a_{12}^{-2} + \frac{1}{\Gamma^2} a_{11}^x a_{11}^{-1} a_{12}^{-1}. \end{aligned}$$

By Part (2) of Remark 7,

$$(\Lambda \Phi_{\beta}^{-L} A)_{11}^{xy} - A_{11}^{xy} = \lambda(\mu^2 a_{11}^{x+y-2} - \frac{\mu}{\Gamma} a_{11}^{x-1} a_{11}^{y-1} - \frac{\mu}{\Gamma} a_{11}^x a_{11}^{y-2} + \frac{1}{\Gamma^2} a_{11}^x a_{11}^{-1} a_{11}^{y-1}) - a_{11}^{x+y}.$$

Similarly,

$$(\Lambda \Phi_{\beta}^{-R} \Lambda^{-1})_{11}^{xy} - A_{11}^{xy} = (\lambda \mu^2)^{-1} (a_{11}^{x+y+2} - \frac{1}{\Gamma} a_{11}^{x+1} a_{11}^{y+1} - \frac{1}{\Gamma} a_{11}^{x+2} a_{11}^y + \frac{1}{\Gamma^2} a_{11}^{x+1} a_{11}^1 a_{11}^y) - a_{11}^{x+y}.$$

Since we have $a_{11}^{x+2} - \lambda \mu^2 a_{11}^x$, then the above two relations can be simplified as

$$a_{11}^{x-1} a_{11}^{y-1} + a_{11}^x a_{11}^{y-2} - \frac{1}{\Gamma \mu} a_{11}^x a_{11}^{-1} a_{11}^{y-1} \text{ and}$$

$$a_{11}^{x-1} a_{11}^{y-1} + a_{11}^x a_{11}^{y-2} - \frac{1}{\Gamma} a_{11}^{x-1} a_{11}^1 a_{11}^{y-2}.$$

And only parities of x and y will make a difference in the above two relations.

Direct calculation shows that $HC_0(\beta) \simeq R[X]/\langle (1-\mu)X, X^2 - \Gamma^2 \lambda(1+\mu)^2 \rangle$. Replacing λ by $\lambda \mu^{-f}$, we obtain $HC_0(\beta; f)$.

It will be shown in Section 4.2 that $\widehat{\alpha_0^2}$ is a particular knot in a large family of knots, namely the torus knots. Explicitly, it is the $(1, 2)$ -knot. See Section 4.2 for a definition of torus knots and more examples.

3.3. Properties of $\Phi^{\pm L}, \Phi^{\pm R}$. We give several propositions which will be used in proving the invariance of $HC_0(K; l)$. A similar version of these propositions are proved in [6] where the author defined the HC_0 invariant for knots in S^3 .

If ϕ is an algebra morphism from \mathcal{A}_n to \mathcal{A}_n , and $M \in M_n(M_{\infty}(\mathcal{A}_n))$, we denote by $\phi(M)$ or $M(\phi)$ the matrix obtained from M by replacing each a_{ij}^x by $\phi(a_{ij}^x)$. Recall in last subsection, we defined the four matrices $\Phi_{\beta}^{-L}, \Phi_{\beta}^{-R}, \Phi_{\beta}^{+L}, \Phi_{\beta}^{+R} \in M_n(M_{\infty}(\mathcal{A}_n))$ for $\beta \in \mathcal{C}_n$.

Proposition 3. Let $\beta_1, \beta_2 \in \mathcal{C}_n$ be two braids, then we have

$$\Phi_{\beta_1 \beta_2}^{-L} = \Phi_{\beta_2}^{-L} (\Phi_{\beta_1}) \Phi_{\beta_1}^{-L}$$

$$\Phi_{\beta_1 \beta_2}^{-R} = \Phi_{\beta_1}^{-R} \Phi_{\beta_2}^{-R} (\Phi_{\beta_1})$$

$$\Phi_{\beta_1 \beta_2}^{+L} = \Phi_{\beta_2}^{+L} (\Phi_{\beta_1}) \Phi_{\beta_1}^{+L}$$

$$\Phi_{\beta_1 \beta_2}^{+R} = \Phi_{\beta_1}^{+R} \Phi_{\beta_2}^{+R} (\Phi_{\beta_1})$$

Proof The proof of the four equalities are straight forward and completely analogous, so we will just prove the first one.

By definition, $\Phi_{\beta_2}^{-}(a_{i,n+1}^x) = \sum_{k=1}^n \sum_{z \in \mathbb{Z}} (\Phi_{\beta_2}^{-L})_{ik}^{xz} a_{k,n+1}^z$. So

$$\Phi_{\beta_1 \beta_2}^{-}(a_{i,n+1}^x) = \Phi_{\beta_1}^{-} \Phi_{\beta_2}^{-}(a_{i,n+1}^x)$$

$$\begin{aligned}
&= \sum_{k=1}^n \sum_{z \in \mathbb{Z}} \Phi_{\beta_1}^- ((\Phi_{\beta_2}^{-L})^{xz}) \Phi_{\beta_1}^- (a_{k,n+1}^z) \\
&= \sum_{k,j=1}^n \sum_{z,y \in \mathbb{Z}} \Phi_{\beta_2}^{-L} (\Phi_{\beta_1})^{xz} (\Phi_{\beta_1}^{-L})^{zy} a_{j,n+1}^y \\
&= \sum_{j=1}^n \sum_{y \in \mathbb{Z}} (\Phi_{\beta_2}^{-L} (\Phi_{\beta_1}) \Phi_{\beta_1}^{-L})^{xy} a_{j,n+1}^y
\end{aligned}$$

On the other hand, by definition, $\Phi_{\beta_1\beta_2}^- (a_{i,n+1}^x) = \sum_{j=1}^n \sum_{z \in \mathbb{Z}} (\Phi_{\beta_1\beta_2}^{-L})^{xy} a_{j,n+1}^y$.

Therefore, we have $(\Phi_{\beta_2}^{-L} (\Phi_{\beta_1}) \Phi_{\beta_1}^{-L})^{xy} = (\Phi_{\beta_1\beta_2}^{-L})^{xy}$. \square

Let $I_n \in M_n(M_\infty(\mathcal{A}_n))$ be the identity matrix, i.e. $(I_n)_{ij}^{xy} = \delta_{i,j} \delta_{x,y}$. Then apparently, for a trivial braid $\beta \in \mathcal{C}_n$, $\Phi_\beta^{-L}, \Phi_\beta^{-R}, \Phi_\beta^{+L}, \Phi_\beta^{+R}$ are all equal to I_n . Therefore, we have the following corollary.

Corollary 3. For any braid $\beta \in \mathcal{C}_n$, $\Phi_\beta^{-L}, \Phi_\beta^{-R}, \Phi_\beta^{+L}, \Phi_\beta^{+R}$ are all invertible. Explicitly,

$$\begin{aligned}
(\Phi_\beta^{-L})^{-1} &= \Phi_{\beta^{-1}}^{-L}(\Phi_\beta), & (\Phi_\beta^{-R})^{-1} &= \Phi_{\beta^{-1}}^{-R}(\Phi_\beta), \\
(\Phi_\beta^{+L})^{-1} &= \Phi_{\beta^{-1}}^{+L}(\Phi_\beta), & (\Phi_\beta^{+R})^{-1} &= \Phi_{\beta^{-1}}^{+R}(\Phi_\beta).
\end{aligned}$$

Proof In Proposition 3, set $\beta_1 = \beta, \beta_2 = \beta^{-1}$. \square

Proposition 4. For any $\beta \in \mathcal{C}_n$, we have $\Phi_\beta(A) = \Phi_\beta^{-L} A \Phi_\beta^{-R} = \Phi_\beta^{+L} A \Phi_\beta^{+R}$.

Proof By Proposition 3, it suffices to show the above equation holds for any $\alpha_k \in \mathcal{C}_n$. This can be verified directly, though maybe tediously.

Here we provide another way to prove it.

$$\begin{aligned}
&\Phi_\beta(A_{ij}^{xy}) = \Phi_\beta(a_{ij}^{x+y}) = \Phi_\beta(a_{i,n+1}^x * a_{n+1,j}^y) \\
&\stackrel{\text{Proposition 2}}{=} \Phi_\beta^-(a_{i,n+1}^x) * \Phi_\beta^-(a_{n+1,j}^y) \\
&= \left(\sum_{k=1}^n \sum_{z \in \mathbb{Z}} (\Phi_\beta^{-L})^{xz} a_{k,n+1}^z \right) * \left(\sum_{k'=1}^n \sum_{z' \in \mathbb{Z}} a_{n+1,k'}^{z'} (\Phi_\beta^{-R})^{z'y} \right) \\
&= \sum_{k,k'=1}^n \sum_{z,z' \in \mathbb{Z}} (\Phi_\beta^{-L})^{xz} a_{kk'}^{z+z'} (\Phi_\beta^{-R})^{z'y} \\
&= \sum_{k,k'=1}^n \sum_{z,z' \in \mathbb{Z}} (\Phi_\beta^{-L})^{xz} A_{kk'}^{zz'} (\Phi_\beta^{-R})^{z'y} = (\Phi_\beta^{-L} A \Phi_\beta^{-R})_{ij}^{xy}
\end{aligned}$$

The other equation can be proved analogously. \square

Corollary 4. For $\beta \in \mathcal{C}_n$, $1 \leq p, q \leq n$, $f \in \mathbb{Z}$, the elements of $A - \Lambda_{f;p,q} \Phi_\beta(A) \Lambda_{f;p,q}^{-1}$ are in $\mathcal{I}_{\beta;f;p,q}$. More generally, if $b = a_{i_1, i_2}^{x_1} a_{i_2, i_3}^{x_2} \cdots a_{i_k, i_{k+1}}^{x_k}$, $c_i = \lambda^{\delta_{i,p}} \mu^{-f \delta_{i,q}}$, then $b - c_{i_1} \Phi_\beta(b) c_{i_{k+1}}^{-1}$ is in $\mathcal{I}_{\beta;f;p,q}$.

Proof Set $\Lambda = \Lambda_{f;p,q}$. Then

$$A - \Lambda \Phi_\beta(A) \Lambda^{-1} = A - \Lambda \Phi_\beta^{-L} A \Phi_\beta^{-R} \Lambda^{-1} = A - \Lambda \Phi_\beta^{-L} A + \Lambda \Phi_\beta^{-L} (A - A \Phi_\beta^{-R} \Lambda^{-1}).$$

The elements of the right hand side of the above equation are in $\mathcal{I}_{\beta;f;p,q}$, which implies the first part of the corollary. The more general statement in the corollary is then a direct consequence. \square

3.4. Invariance proof. In this subsection, we prove Theorem 5. Apparently, the three parts in the theorem correspond to the three types of Markov moves introduced in Theorem 1. In the following three subsections, we prove each part of the theorem, respectively.

3.4.1. Invariance under Markov Move I. Let $\tilde{\beta} = \alpha^{-1} \beta \alpha$, $\alpha, \beta \in \mathcal{C}_n$, $f \in \mathbb{Z}$, and assume $\alpha(m) = 1$. Set $\Lambda_i = \Lambda_{f;i,i}$. We define an isomorphism $\varphi : HC_0(\tilde{\beta}; f; m) \rightarrow HC_0(\beta; f; 1)$ by specifying the image of the generators.

$$\varphi(A) := \Phi_\alpha(A), \text{ i.e. } \varphi(a_{ij}^x) := \Phi_\alpha(a_{ij}^x)$$

We need to show $\varphi(\mathcal{I}_{\tilde{\beta};f;m,m}) \subset \mathcal{I}_{\beta;f;1,1}$.

First of all, by using Proposition 3, we have

$$\Phi_\alpha(\Phi_{\alpha^{-1}\beta\alpha}^{-L}) = \Phi_\alpha(\Phi_{\beta\alpha}^{-L}(\Phi_{\alpha^{-1}}) \Phi_{\alpha^{-1}}^{-L}) = \Phi_{\beta\alpha}^{-L} \Phi_{\alpha^{-1}}^{-L}(\Phi_\alpha) = \Phi_\alpha^{-L}(\Phi_\beta) \Phi_\beta^{-L} \Phi_{\alpha^{-1}}^{-L}(\Phi_\alpha),$$

Therefore, we have

$$\begin{aligned} \varphi(A - \Lambda_m \Phi_{\tilde{\beta}}^{-L} A) &= \varphi(A) - \Lambda_m \varphi(\Phi_{\tilde{\beta}}^{-L}) \varphi(A) \\ &= \Phi_\alpha(A) - \Lambda_m \Phi_\alpha(\Phi_{\alpha^{-1}\beta\alpha}^{-L}) \Phi_\alpha(A) \\ &= \Phi_\alpha(A) - \Lambda_m \Phi_\alpha^{-L}(\Phi_\beta) \Phi_\beta^{-L} \Phi_{\alpha^{-1}}^{-L}(\Phi_\alpha) \Phi_\alpha^{-L} A \Phi_\alpha^{-R} \\ &= \Phi_\alpha^{-L} A \Phi_\alpha^{-R} - \Lambda_m \Phi_\alpha^{-L}(\Phi_\beta) \Phi_\beta^{-L} A \Phi_\alpha^{-R} \\ &= (\Phi_\alpha^{-L} - \Lambda_m \Phi_\alpha^{-L}(\Phi_\beta) \Lambda_1^{-1}) A \Phi_\alpha^{-R} + \\ &\quad \Lambda_m \Phi_\alpha^{-L}(\Phi_\beta) \Lambda_1^{-1} (A - \Lambda_1 \Phi_\beta^{-L} A) \Phi_\alpha^{-R} \end{aligned}$$

Since $(\Phi_\alpha^{-L})_{ij}^{xy}$ is a non-commutative polynomial in which each monomial is of the form $a_{\alpha(i), j_1}^{x_1} a_{j_1, j_2}^{x_2} \cdots a_{j_{k-1}, j}^{x_k}$, and note that $\delta_{i,m} = \delta_{\alpha(i), 1}$, then $(\Phi_\alpha^{-L} - \Lambda_m \Phi_\alpha^{-L}(\Phi_\beta) \Lambda_1^{-1})_{ij}^{xy}$ is a sum of polynomials of the form $a_{\alpha(i), j_1}^{x_1} a_{j_1, j_2}^{x_2} \cdots a_{j_{k-1}, j}^{x_k} - (\lambda \mu^{-f})^{\delta_{\alpha(i), 1}} \Phi_\beta(a_{\alpha(i), j_1}^{x_1} a_{j_1, j_2}^{x_2} \cdots a_{j_{k-1}, j}^{x_k}) (\lambda \mu^{-f})^{-\delta_{j, 1}}$, which, by Corollary 4, is in $\mathcal{I}_{\beta;f;1,1}$.

Since elements of $A - \Lambda_1 \Phi_{\tilde{\beta}}^{-L} A$ are also in $\mathcal{I}_{\beta;f;1,1}$, this implies $\varphi(A - \Lambda_m \Phi_{\tilde{\beta}}^{-L} A) \subset \mathcal{I}_{\beta;f;1,1}$.

The proofs of the other three relations $\varphi(A - A\Phi_{\tilde{\beta}}^{-R}\Lambda_m^{-1})$, $\varphi(A - \Lambda_m \Phi_{\tilde{\beta}}^{+L} A)$, $\varphi(A - A\Phi_{\tilde{\beta}}^{+R}\Lambda_m^{-1})$ are completely analogous.

This shows $\varphi(\mathcal{I}_{\tilde{\beta}}; f; m, m) \subset \varphi(\mathcal{I}_{\beta}; f; 1, 1)$ and thus induces a well-defined map $HC_0(\tilde{\beta}; f) \longrightarrow HC_0(\beta; f)$. In a similar way, we can define the inverse map $HC_0(\beta; f) \longrightarrow HC_0(\tilde{\beta}; f)$ by sending A to $\Phi_{\alpha^{-1}}(A)$ and show that it is well defined. Thus φ is an isomorphism.

3.4.2. Invariance under Markov Move II. For any $\beta \in \mathcal{C}_n$, $f \in \mathbb{Z}$ let $\tilde{\beta} = \epsilon^{-}(\beta)\alpha_n$. We show $HC_0(\tilde{\beta}; f) \simeq HC_0(\beta; f+1)$.

Remark 9. The proof of $HC_0(\epsilon^{-}(\beta)\alpha_n^{-1}; f+1) \simeq HC_0(\beta; f)$ is completely analogous. To save space, we omit its proof here.

Define $\varphi : HC_0(\tilde{\beta}; f; n+1) \longrightarrow HC_0(\beta; f+1; n)$,

$$(3.1) \quad \varphi(a_{ij}^x) = \begin{cases} a_{nn}^x & i = n+1, j = n+1 \\ \mu a_{nj}^x & i = n+1, j \leq n \\ \mu^{-1} a_{in}^x & i \leq n, j = n+1 \\ a_{ij}^x & i \leq n, j \leq n \end{cases}$$

The verification that φ maps $\mathcal{I}_{\tilde{\beta};f;n+1,n+1}$ to $\mathcal{I}_{\beta;f+1;n,n}$ consists of direct but long calculations. we will only show $\varphi(a_{i,j}^{x+y} - (\lambda\mu^{-f})^{\delta_{i,n+1}} \Phi_{\tilde{\beta}}^{-}(a_{i,n+2}^x) * a_{n+2,j}^y) \in \mathcal{I}_{\beta;f+1;n,n}$. The other relations can be proven similarly.

Set $c = \lambda\mu^{-f}$, $\mathcal{I} = \mathcal{I}_{\beta;f+1;n,n}$.

Case 1 : $i = n+1, j \leq n$.

$$\begin{aligned} \varphi(a_{n+1,j}^{x+y} - c\Phi_{\tilde{\beta}}^{-}(a_{n+1,n+2}^x) * a_{n+2,j}^y) &= \mu a_{n,j}^{x+y} - c\varphi(\Phi_{\epsilon^{-}(\beta)}^{-}(a_{n,n+2}^x) * a_{n+2,j}^y) \\ &= \mu a_{n,j}^{x+y} - c\Phi_{\beta}^{-}(a_{n,n+1}^x) * a_{n+1,j}^y = \mu(a_{n,j}^{x+y} - \lambda\mu^{-f-1}\Phi_{\beta}^{-}(a_{n,n+1}^x) * a_{n+1,j}^y) \in \mathcal{I} \end{aligned}$$

Note that here we used the fact that $\Phi_{\epsilon^{-}(\beta)}^{-}(a_{n,n+2}^x) * a_{n+2,j}^y = \Phi_{\beta}^{-}(a_{n,n+1}^x) * a_{n+1,j}^y \in \mathcal{A}_{n+1}$.

Case 2 : $i = n, j \leq n$.

$$\begin{aligned} &a_{n,j}^{x+y} - \Phi_{\tilde{\beta}}^{-}(a_{n,n+2}^x) * a_{n+2,j}^y \\ &= a_{n,j}^{x+y} - \Phi_{\epsilon^{-}(\beta)}^{-}(-a_{n+1,n+2}^x + \frac{1}{\Gamma\mu} a_{n+1,n}^0 a_{n,n+2}^x) * a_{n+2,j}^y \\ &= a_{n,j}^{x+y} - (-a_{n+1,n+2}^x + \frac{1}{\Gamma\mu} \Phi_{\epsilon^{-}(\beta)}^{-}(a_{n+1,n}^0) \Phi_{\epsilon^{-}(\beta)}^{-}(a_{n,n+2}^x)) * a_{n+2,j}^y \\ &= a_{n,j}^{x+y} + a_{n+1,j}^{x+y} - \frac{1}{\Gamma\mu} \Phi_{\epsilon^{-}(\beta)}^{-}(a_{n+1,n}^0) \Phi_{\epsilon^{-}(\beta)}^{-}(a_{n,n+2}^x) * a_{n+2,j}^y \\ &= a_{n,j}^{x+y} + a_{n+1,j}^{x+y} - \frac{1}{\Gamma\mu} \Phi_{\beta}^{-}(a_{n+1,n}^0) \Phi_{\beta}^{-}(a_{n,n+1}^x) * a_{n+1,j}^y \end{aligned}$$

Since $\varphi(\Phi_{\beta}^{-}(a_{n+1,n}^0)) = \mu a_{n,n+1}^0 * \Phi_{\beta}^{-}(a_{n+1,n}^0) = c a_{nn}^0 \pmod{\mathcal{I}}$,

$$\begin{aligned}
 & \varphi(a_{n,j}^{x+y} - \Phi_{\tilde{\beta}}^-(a_{n,n+2}^x) * a_{n+2,j}^y) \\
 &= (1+u)a_{n,j}^{x+y} - \frac{1}{\Gamma\mu}c(1+u)\Gamma\Phi_{\tilde{\beta}}^-(a_{n,n+1}^x) * a_{n+1,j}^y \pmod{\mathcal{I}} \\
 &= (1+u)(a_{n,j}^{x+y} - \lambda\mu^{-f-1}\Phi_{\tilde{\beta}}^-(a_{n,n+1}^x) * a_{n+1,j}^y) \pmod{\mathcal{I}} = 0 \pmod{\mathcal{I}}.
 \end{aligned}$$

Case 3 : $i \leq n-1, j \leq n$.

$$\begin{aligned}
 & \varphi(a_{i,j}^{x+y} - \Phi_{\tilde{\beta}}^-(a_{i,n+2}^x) * a_{n+2,j}^y) = \varphi(a_{i,j}^{x+y} - \Phi_{\epsilon^-(\beta)}^-(a_{i,n+2}^x) * a_{n+2,j}^y) \\
 &= \varphi(a_{i,j}^{x+y} - \Phi_{\beta}^-(a_{i,n+1}^x) * a_{n+1,j}^y) = a_{i,j}^{x+y} - \Phi_{\beta}^-(a_{i,n+1}^x) * a_{n+1,j}^y \in \mathcal{I}.
 \end{aligned}$$

Case 4 : $j = n+1$. The proof is the same as the above three cases except an overall scalar μ^{-1} is multiplied to each expression.

This finishes the verification. One can also define a map $HC_0(\beta; f+1; n) \rightarrow HC_0(\tilde{\beta}; f; n+1)$ sending a_{ij}^x to a_{ij}^x , and show that it is well defined. Clearly this is the inverse of φ .

3.4.3. Invariance under Markov move III. Recall that D_n is the unit disk with $n+1$ punctures p, p_1, \dots, p_n centered at the origin of the complex plane. To be more precise, let p be the origin and the coordinate of p_i be $\frac{i}{n+1}$. We define a map $r : D_n \rightarrow D_n$ by $r(z) = \frac{\bar{z}}{|z|} - \bar{z}$. Namely, r is a reflection about the x -axis followed by another reflection about the circle centered at the origin with radius $\frac{1}{2}$. Note that $r^2 = Id$. Also $r \times Id$ defines a map on $X = D_n \times [0, 1]$, which will still be denoted by r .

Recall that \mathcal{C}_n is the braid group on the punctured disk D_n inside X . Therefore, r induces a group isomorphism from \mathcal{C}_n to itself. Explicitly, the isomorphism, also denote by r , is given by:

$$(3.2) \quad r(\alpha_i) = \begin{cases} (\alpha_{n-1} \cdots \alpha_1 \alpha_0 \alpha_1 \cdots \alpha_{n-1})^{-1} & i = 0 \\ \alpha_{n-i} & 1 \leq i \leq n-1 \end{cases}$$

Lemma 4. The map r defined above from \mathcal{C}_n to \mathcal{C}_n is a group isomorphism and $r^2 = Id$.

Proof This can be verified purely algebraically. □

Also recall that q_1, \dots, q_n are n points with the coordinate $\frac{i}{n+1} - \epsilon$ for some tiny $\epsilon > 0$. And $Q_n = \{q_i, 1 \leq i \leq n\}$, $\mathcal{Q}_n = \{\gamma : [0, 1] \rightarrow D_n \mid \gamma \text{ is continuous, } \gamma(0), \gamma(1) \in Q_n\} / \sim$. Let $q'_{n+1-i} = r(q_i)$, which has the coordinate $\frac{n+1-i}{n+1} + \epsilon$, and let $Q'_n = \{q'_i, 1 \leq i \leq n\}$. It should be clear that in the definition of $\tilde{\mathcal{A}}_n$, if we replace q_i by q'_i , insist that the curves start and end at q'_i , and change the skein relations accordingly, then we get the same algebra.

For a curve $\gamma \in \mathcal{Q}_n$ from q_i to q_j , $r(\gamma)$ is a curve from q'_{n+1-i} to q'_{n+1-j} . And it's not hard to check that this map also preserves the

“skein” relations in Figure 5 that defines \tilde{A}_n . Thus r induces an algebra isomorphism from \tilde{A}_n to \tilde{A}_n .

Explicitly, the map $r : \tilde{A}_n \longrightarrow \tilde{A}_n$ is given by Figure 12.

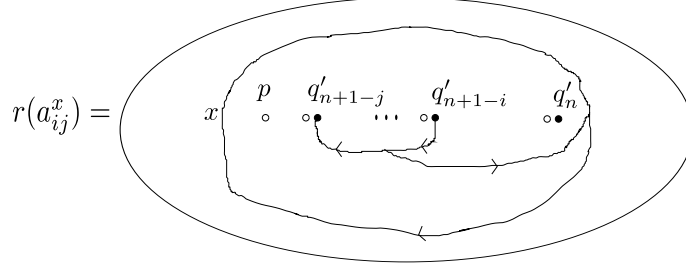


FIGURE 12. $r(a_{ij}^x)$

Remark 10. r also extends to a bijection from \mathcal{Q}_n^+ to \mathcal{Q}_n^- by furthermore requiring that p_0 is mapped to p_{n+1} . And r maps the “skein” relations that define \mathcal{A}_n^+ to the corresponding “skein” relations that define \mathcal{A}_n^- . Consequently, we get an isomorphism $r : \mathcal{A}_n^+ \longrightarrow \mathcal{A}_n^-$. Note that the inverse map is also induced by r that maps \mathcal{Q}_n^- to \mathcal{Q}_n^+ . For this reason, we will denote the inverse map also by r . In summary, r is an isomorphism between \mathcal{A}_n^+ and \mathcal{A}_n^- , which restricts to an isomorphism on \mathcal{A}_n and which has square Id .

Lemma 5. If $P, Q \in \mathcal{A}_n^\pm$ are connectable, then $r(P), r(Q)$ are connectable, and $r(P * Q) = r(P) * r(Q)$.

Proof Clear from the geometrical interpretation of a_{ij}^x and the map r . \square

Lemma 6. If β is a braid in \mathcal{C}_n , then we have $r \circ \Phi_\beta = \Phi_{r(\beta)} \circ r$. More generally, we have $r \circ \Phi_\beta^- = \Phi_{r(\beta)}^+ \circ r$.

Proof It’s possible, though tedious, to prove it algebraically. For example, it suffices to prove the case for $\beta = \alpha_k^{\pm 1}$ acting on a_{ij}^x . Here we give another geometric proof which makes the statement in the lemma almost trivial. Recall that the isomorphism $r : \mathcal{C}_n \longrightarrow \mathcal{C}_n$ is induced by the homeomorphism $r \times Id : D_n \times I \longrightarrow D_n \times I$. By Remark 5, $\beta(\gamma_{ij}^x)$ can be obtained as the curve by sliding γ_{ij}^x in $D_n \times \{0\}$ along the parallel copy braid diagram β' up to $D_n \times \{1\}$. The map $r \times Id$ maps γ_{ij}^x to $r(\gamma_{ij}^x)$, $\Phi_\beta(\gamma_{ij}^x)$ to $r \circ \Phi_\beta(\gamma_{ij}^x)$, and β to $r(\beta)$. Thus $r \circ \Phi_\beta(\gamma_{ij}^x)$ is obtained by sliding $r(\gamma_{ij}^x)$ along the parallel copy braid diagram $r(\beta)'$, and therefore $\Phi_{r(\beta)} \circ r(\gamma_{ij}^x) = r \circ \Phi_\beta(\gamma_{ij}^x)$.

The more general equation can be proved analogously by using Remark 5 and Lemma 7. \square

Theorem 6. For $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$ the map $r : \mathcal{A}_n \longrightarrow \mathcal{A}_n$ induces an isomorphism from $HC_0(\beta; f; 1)$ to $HC_0(r(\beta); f; n)$.

Proof It suffices to show r maps $I_{\beta; f; 1, 1}$ to $I_{r(\beta); f; n, n}$. Set $c = \lambda\mu^{-f}$.
 $r((\Lambda_{f; 1, 1} \Phi_{\beta}^{-L} A)_{ij}^{xy}) = r(c^{\delta_{i, 1}} \Phi_{\beta}^{-}(a_{i, n+1}^x) * a_{n+1, j}^y) = c^{\delta_{i, 1}} (r \circ \Phi_{\beta}^{-}(a_{i, n+1}^x)) * r(a_{n+1, j}^y) = c^{\delta_{n+1-i, n}} (\Phi_{r(\beta)}^{+} \circ r(a_{i, n+1}^x)) * r(a_{n+1, j}^y)$.

The first identity in the above equation is by the argument in Part (2) of Remark 7, the second identity is by Lemma 5, and the third by Lemma 6.

Assume $r(a_{i, n+1}^x) = \sum P_k^z a_{k0}^{z'}$, $r(a_{n+1, j}^y) = \sum a_{0k'}^{z'} Q_{k'}^{z'}$, where $P_k^z, Q_{k'}^{z'}$ are elements in \mathcal{A}_n . Then

$$\begin{aligned} & r((A - \Lambda_{f; 1, 1} \Phi_{\beta}^{-L} A)_{ij}^{xy}) \\ &= \sum P_k^z a_{kk'}^{zz'} Q_{k'}^{z'} - c^{\delta_{n+1-i, n}} \Phi_{r(\beta)}^{-}(P_k^z) \Phi_{r(\beta)}^{+}(a_{k0}^{z'}) * a_{0k'}^{z'} Q_{k'}^{z'} \\ &= \sum (P_k^z - c^{\delta_{n+1-i, n}} \Phi_{r(\beta)}^{-}(P_k^z) c^{-\delta_{k, n}}) a_{kk'}^{zz'} Q_{k'}^{z'} + c^{\delta_{n+1-i, n}} \Phi_{r(\beta)}^{-}(P_k^z) c^{-\delta_{k, n}} (a_{kk'}^{zz'} - c^{\delta_{k, n}} \Phi_{r(\beta)}^{+}(a_{k0}^{z'}) * a_{0k'}^{z'}) Q_{k'}^{z'} \end{aligned}$$

Note that P_k^z is a sum of monomials of the form $a_{n+1-i, i_1}^{x_1} a_{i_1, i_2}^{x_2} \cdots a_{i_{m-1}, k}^{x_m}$, then $P_k^z - c^{\delta_{n+1-i, n}} \Phi_{r(\beta)}^{-}(P_k^z) c^{-\delta_{k, n}}$ is in $I_{r(\beta); f; n, n}$ by Corollary 4.

Then it follows that $r((A - \Lambda_{f; 1, 1} \Phi_{\beta}^{-L} A)_{ij}^{xy})$ is in $I_{r(\beta); f; n, n}$.

The other relations are proved in basically the same way. And thus we showed r is well-defined. The fact that r is an isomorphism is trivial to check. □

Now we prove $HC_0(\beta; f)$ is invariant under Markov move III. A key observation is the following commuting diagram.

$$(3.3) \quad \begin{array}{ccc} \mathcal{C}_n & \xrightarrow{\epsilon^+} & \mathcal{C}_{n+1} \\ \downarrow r & & \downarrow r \\ \mathcal{C}_n & \xrightarrow{\epsilon^-} & \mathcal{C}_{n+1} \end{array}$$

Lemma 7. The above diagram commutes, namely $r \circ \epsilon^+ = \epsilon^- \circ r : \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$.

Proof We only need to check on the generators.

$$\begin{aligned} r\epsilon^+(\alpha_0) &= r(\alpha_1 \alpha_0 \alpha_1) = \alpha_n (\alpha_n \cdots \alpha_1 \alpha_0 \alpha_1 \cdots \alpha_n)^{-1} \alpha_n \\ &= (\alpha_{n-1} \cdots \alpha_1 \alpha_0 \alpha_1 \cdots \alpha_{n-1})^{-1} = \epsilon^- r(\alpha_0). \end{aligned}$$

$$\text{For } i \geq 1, r\epsilon^+(\alpha_i) = r(\alpha_{i+1}) = \alpha_{n-i} = \epsilon^- r(\alpha_i) \quad \square$$

Let $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$, then $r(\epsilon^+(\beta) \alpha_1^{\pm 1}) = r(\epsilon^+(\beta)) r(\alpha_1^{\pm 1}) = \epsilon^-(r(\beta)) \alpha_n^{\pm 1}$. Therefore,

$$\begin{aligned} HC_0(\epsilon^+(\beta) \alpha_1^{\pm 1}; f) &\simeq HC_0(r(\epsilon^+(\beta) \alpha_1^{\pm 1}); f) = HC_0(\epsilon^-(r(\beta)) \alpha_n^{\pm 1}; f) \\ &\simeq HC_0(r(\beta); f \pm 1) \simeq HC_0(\beta; f \pm 1). \end{aligned}$$

The first and last isomorphism above are due to Theorem 6 and the second isomorphism is the invariance isomorphism under Markov move II.

Now we finished showing $HC_0(\beta; f)$ is invariant under Markov move III.

4. PROPERTIES OF THE INVARIANT

4.1. Symmetries of the invariant. In Section 3.4.3, we already proved that for a braid $\beta \in \mathcal{C}_n$, we have $HC_0(\beta; f) \simeq HC_0(r(\beta); f)$. Here we show the relation between $HC_0(\beta; f)$ and $HC_0(\beta^{-1}; f)$.

Proposition 5. Let $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$, then $HC_0(\beta^{-1}; f)$ is isomorphic to $HC_0(\beta; -f)$ with λ replaced by λ^{-1} .

Proof Let $HC'_0(\beta; -f)$ be the algebra obtained from $HC_0(\beta; -f)$ by replacing λ by λ^{-1} . We define the isomorphism $HC_0(\beta^{-1}; f) \longrightarrow HC'_0(\beta; -f)$ to be the one induced by Φ_β . We need to check Φ_β maps $\mathcal{I}_{\beta^{-1}; f; 1, 1}$ to $\mathcal{I}_{\beta; -f; 1, 1}$ with λ replaced by λ^{-1} . Set $\Lambda = \Lambda_{f; 1, 1}$, and note that Λ^{-1} is exactly the matrix $\Lambda_{-f; 1, 1}$ with λ replaced by λ^{-1} .

$$\begin{aligned} \Phi_\beta(\Lambda \Phi_{\beta^{-1}}^{+L} A - A) &= \Lambda \Phi_{\beta^{-1}}^{+L} (\Phi_\beta) \Phi_\beta(A) - \Phi_\beta(A) = \Lambda \Phi_{\beta^{-1}}^{+L} (\Phi_\beta) \Phi_\beta^{+L} A \Phi_\beta^{+R} - \\ &\Phi_\beta^{+L} A \Phi_\beta^{+R} = \Lambda A \Phi_\beta^{+R} - \Phi_\beta^{+L} A \Phi_\beta^{+R} = \Lambda(A - \Lambda^{-1} \Phi_\beta^{+L} A) \Phi_\beta^{+R} \end{aligned}$$

The second equality is by Proposition 4 and the third one is by Corollary 3.

The other three relations can be proved analogously. Therefore, Φ_β induces a well-defined algebra map from $HC_0(\beta^{-1}; f)$ to $HC'_0(\beta; -f)$. It's easy to check it's also an isomorphism. \square

4.2. Torus Knots. In this subsection we study some properties of the torus knots in $S^1 \times S^2$.

Let C be the equator of S^2 , then $S^1 \times C$ is a torus which bounds two solid tori in $S^1 \times S^2$, with $z_0 \times C$ being the meridian and $S^1 \times z_1$ the longitude. In [1], a knot in $S^1 \times S^2$ is called a torus knot if it can be isotoped to a knot in $S^1 \times C$. Fix a meridian M and a longitude L in $S^1 \times C$, and let p, q be two relatively prime integers. A (p, q) -knot in $S^1 \times S^2$ is a knot which can be isotoped to $pM + qL$ in $S^1 \times C$. In general, for a knot K and a framing l , $HC_0(K; l)$ may not be finitely generated as an R -algebra. However, we show below that for torus knots, the invariant indeed is always finitely generated.

Theorem 7. Let K be a (p, q) -knot in $S^1 \times S^2$ with framing l where p, q are relatively prime integers, then $HC_0(K; l)$ is finitely generated as an R -algebra. Moreover, the minimum number of algebra generators is no more than $q - 1$.

Proof By Remark 2, a (p, q) -knot is represented by the braid $\beta(p, q) = (\alpha_0 \cdots \alpha_{p-1})^q$. See Figure 13 for a picture of $(3, 2)$ -knot. For simplicity, we still use β to denote $\beta(p, q)$. Also for reasons that will become clear below, we use the notation $b_{ij}^x = a_{i+1, j+1}^x$. Assume $HC_0(K; l) = HC_0(\beta; f) = \mathcal{A}_p / \mathcal{I}_{\beta; f; 1, 1}$, and set $c = \lambda \mu^{-f}$. It's easy to check that the following equation holds:

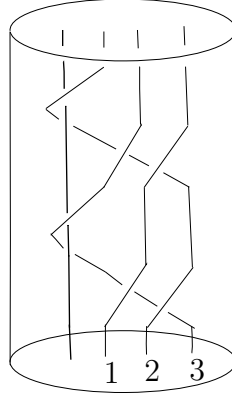


FIGURE 13. $(3, 2)$ -knot

$$(4.1) \quad \Phi_{\beta(p, 1)}^+(a_{i0}^x) = \begin{cases} a_{i+1, 0}^x & 1 \leq i \leq p-1 \\ \mu a_{1, 0}^{x-1} & i = p \end{cases}$$

Then we have $\Phi_{\beta(p, q)}^+(a_{i0}^x) = \mu^{\lfloor \frac{i-1+q}{p} \rfloor} a_{(i-1+q) \pmod{p} + 1, 0}^{x - \lfloor \frac{i-1+q}{p} \rfloor}$. Using b_{ij}^x to replace $a_{i+1, j+1}^x$, we get a simpler expression $\Phi_{\beta(p, q)}^+(b_{i, -1}^x) = \mu^{\lfloor \frac{i+q}{p} \rfloor} b_{(i+q) \pmod{p}, -1}^{x - \lfloor \frac{i+q}{p} \rfloor}$.

Thus by Part 2 of Remark 7, the third relation that defines $\mathcal{I}_{\beta; f; 1, 1}$ is

$$(4.2) \quad b_{ij}^x - \mu^{\lfloor \frac{i+q}{p} \rfloor} c^{\delta_{i, 0}} b_{(i+q) \pmod{p}, j}^{x - \lfloor \frac{i+q}{p} \rfloor}, \quad 0 \leq i, j \leq p-1, x \in \mathbb{Z}.$$

Similarly, the fourth relation that defines $\mathcal{I}_{\beta; f; 1, 1}$ is

$$(4.3) \quad b_{ij}^x - \mu^{-\lfloor \frac{j+q}{p} \rfloor} c^{-\delta_{j, 0}} b_{i, (j+q) \pmod{p}}^{x + \lfloor \frac{j+q}{p} \rfloor}, \quad 0 \leq i, j \leq p-1, x \in \mathbb{Z}.$$

Define $g(i, k) := \sum_{r=0}^{k-1} \lfloor \frac{(i+rq) \pmod{p} + q}{p} \rfloor$, $h(i, k) := \sum_{r=0}^{k-1} \delta_{(i+rq) \pmod{p}, 0}$, $0 \leq i \leq p-1, k \geq 1$, and define $f(i, 0) := 0, h(i, 0) := 0$.

It's elementary to check that $g(i, k) = \lfloor \frac{k}{p} \rfloor q + g(i, k \bmod p)$ and $h(i, k) = \lfloor \frac{k}{p} \rfloor + h(i, k \bmod p)$, and in $HC_0(\beta; f; 1, 1)$, we have the equalities $b_{ij}^x = \mu^{g(i,k)} c^{h(i,k)} b_{(i+kq) \bmod p, j}^{x-g(i,k)} = \mu^{-g(j,k)} c^{-h(j,k)} b_{i, (j+kq) \bmod p}^{x+g(j,k)}$, $\forall k \geq 0$. Especially, we have $b_{ij}^x = \mu^{g(i,p)} c^{h(i,p)} b_{ij}^{x-g(i,p)} = \mu^q c b_{ij}^{x-q}$, so b_{ij}^x is periodic, up to a scalar, in x with period equal to q .

Let k_1, k_2 be any numbers that satisfy $k_1 q \bmod p = i$, $k_2 q \bmod p = j$, then $b_{ij}^x = \mu^{g(0,k_2)-g(0,k_1)} c^{h(0,k_2)-h(0,k_1)} b_{00}^{x+g(0,k_1)-g(0,k_2)}$, and $b_{00}^{x+q} = \mu^q c b_{00}^x$. Thus all the b_{ij}^x 's are completely determined by $b_{00}^0 = (1 + \mu)\Gamma, b_{00}^1, \dots, b_{00}^{q-1}$ and the condition that $b_{00}^{x+q} = \mu^q c b_{00}^x$. So $HC_0(\beta; f)$ is finitely generated and $\{b_{00}^x, 1 \leq x \leq q-1\}$ is a set of generators. \square

At the end of this subsection, let's compute some examples of torus knots.

Example 2. 1). **(p, 1)-knot.** The $(p, 1)$ -knot is represented by the braid $\alpha_0 \cdots \alpha_{p-1}$. Clearly, by Markov II in Theorem 1, this braid is equivalent to α_0 representing the $(1, 1)$ -knot. Set $\beta = \alpha_0 \in \mathcal{C}_1, \Lambda = \Lambda_{0,1,1}, f = 0$. By Theorem 7, $a_{11}^{x+1} = \lambda \mu a_{11}^x$. Since $a_{11}^0 = (1 + \mu)\Gamma$, we have $a_{11}^x = (1 + \mu)\Gamma(\lambda \mu)^x$.

By definition, $\Phi_\beta^-(a_{12}^x) = -\mu a_{12}^{x-1} + \frac{1}{\Gamma} a_{11}^x a_{12}^{-1}$, thus $(\Lambda \Phi_\beta^{-L} A - A)_{11}^{xy} = -\lambda \mu a_{11}^{x+y-1} + \frac{\lambda}{\Gamma} a_{11}^x a_{12}^{y-1} - a_{11}^{x+y}$. The second relation can be calculated analogously. By using the fact that $a_{11}^x = (1 + \mu)\Gamma(\lambda \mu)^x$, it's easy to see that $HC_0(\alpha_0) \simeq R/\langle \mu^2 - 1 \rangle$.

2). **(p, 2)-knot.** By Proposition 2.2 in [1], all the $(p, 2)$ -knots are equivalent to each other with p odd. This can also be seen directly by Markov moves. Thus, we only need to compute the $(1, 2)$ -knot, which is represented by $\beta = \alpha_0^2$. It was shown in the second example in Section 3.2 that $HC_0(\alpha_0^2) \simeq R[X]/\langle (1 - \mu)X, X^2 - \Gamma^2 \lambda(1 + \mu)^2 \rangle$.

3). **(p, 3)-knot.** Again by Proposition 2.2 in [1], there are two classes of knots of this type. A representative of each class could be chosen as $(1, 3)$ -knot and $(2, 3)$ -knot. Here we only compute $HC_0(\alpha_0^3)$. Since the calculations are not difficult but tedious, we just present the result obtained by computer packages. $HC_0(\alpha_0^3) \simeq R\langle X, Y \rangle / \langle Y^2 - \Gamma \lambda \mu^4(1 + \mu^2)X, X^2 - \Gamma(1 + \mu^{-2})Y, (1 + \mu^2)(XY - YX), -\mu^2 XY + YX + \Gamma^2 \lambda \mu^4(\mu^2 - 1) \rangle$.

4.3. Local knots. Throughout this subsection, we will set $\Gamma = -1$. A knot is called *local* if it is contained in a 3-ball. It's easy to see that a knot in $S^1 \times S^2$ is local if and only if it can be represented as the closure of a braid which doesn't contain α_0 or α_0^{-1} , i.e. a braid in $\mathcal{B}_n = \langle \alpha_1, \dots, \alpha_{n-1} \rangle \subset \mathcal{C}_n$. Note that the braids in \mathcal{B}_n are closed

under the Markov moves given in Theorem 1, and moreover, Markov move III in this case is a consequence of Markov moves I, II. Since Markov moves I, II are just the classical Markov moves for braids in \mathcal{B}_n representing knots in S^3 , we thus have a one-to-one correspondence between knots in S^3 and local knots in $S^1 \times S^2$.

Let $\beta \in \mathcal{B}_n \subset \mathcal{C}_n$, and let $hc_0(\beta)$ denote the 0-th framed knot contact homology in [8]. Then we have the following decompositions for the HC_0 invariant of local knots, which relates our invariant to hc_0 .

Proposition 6. Let $\beta \in \mathcal{B}_n$ be a braid such that its closure is a local knot in $S^1 \times S^2$, then $HC_0(\beta) \simeq hc_0(\beta) + \sum_{0 \neq x \in \mathbb{Z}} H_x$, where all the H_x 's are isomorphic to each other as subalgebras and there is a surjective algebra morphism from H_x to $hc_0(\beta)$. Moreover, for any $0 < m \in \mathbb{Z}$, $\sum_{|x| \leq m} H_x$ is a proper subalgebra of $HC_0(\beta)$.

Proof Set $\Lambda = \Lambda_{0;1,1}$.

It's easy to see from Equation 2.2 that $\Phi_\beta^-(a_{i,n+1}^x) = \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,n+1}^x$ and that $\Phi_\beta^+(a_{i,0}^x) = \Phi_\beta^-(a_{i,n+1}^x) * a_{n+1,0}^0 = \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,0}^x$. Therefore, $(\Phi_\beta^{+L} A)_{ij}^{xy} = \Phi_\beta^+(a_{i,0}^x) * a_{0,j}^y = \Phi_\beta^-(a_{i,n+1}^x) * a_{n+1,j}^y = (\Phi_\beta^{-L} A)_{ij}^{xy}$, i.e. $\Phi_\beta^{+L} A = \Phi_\beta^{-L} A$. Similarly, we have $A \Phi_\beta^{+R} = A \Phi_\beta^{-R}$. So to compute $HC_0(\beta)$, we only need to consider the relations $A - \Lambda \Phi_\beta^{-L} A$, $A - A \Phi_\beta^{-R} \Lambda^{-1}$.

$(\Lambda \Phi_\beta^{-L} A)_{ij}^{xy} = \lambda^{\delta_{i,1}} \Phi_\beta^-(a_{i,n+1}^x) * a_{n+1,j}^y = \lambda^{\delta_{i,1}} \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,j}^{x+y} = \lambda^{\delta_{i,1}} \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,j}^0 * a_{j,j}^{x+y} = (\Lambda \Phi_\beta^{-L} A)_{ij}^{00} * a_{j,j}^{x+y}$.

Therefore, we have $A_{ij}^{xy} - (\Lambda \Phi_\beta^{-L} A)_{ij}^{xy} = (A_{ij}^{00} - (\Lambda \Phi_\beta^{-L} A)_{ij}^{00}) * a_{j,j}^{x+y}$. Similarly, we have $A_{ij}^{xy} - (A \Phi_\beta^{-R} \Lambda^{-1})_{ij}^{xy} = a_{i,i}^{x+y} * (A_{ij}^{00} - (A \Phi_\beta^{-R} \Lambda^{-1})_{ij}^{00})$.

Let $E_0 = \mathbb{Z}\langle a_{ij}^0, 1 \leq i, j \leq n \rangle$ and for $x \neq 0$, $E_x = \mathbb{Z}\langle a_{ij}^0, a_{ij}^x, 1 \leq i, j \leq n \rangle$. For all $x \in \mathbb{Z}$, let $H_x = E_x / \langle (A_{ij}^{00} - (\Lambda \Phi_\beta^{-L} A)_{ij}^{00}) * a_{j,j}^{x+y}, a_{i,i}^x * (A_{ij}^{00} - (A \Phi_\beta^{-R} \Lambda^{-1})_{ij}^{00}), 1 \leq i, j \leq n \rangle$. Then clearly for $x \neq 0, y \neq 0$, H_x is isomorphic to H_y by sending a_{ij}^0 to a_{ij}^0 and a_{ij}^x to a_{ij}^y . And there is also a surjective morphism from H_x to H_0 by sending a_{ij}^0, a_{ij}^x both to a_{ij}^0 . Similarly, for all $x \in \mathbb{Z}$, one can define a surjective morphism from $HC_0(\beta)$ to H_x by sending a_{ij}^x to a_{ij}^x and a_{ij}^z to a_{ij}^0 for all $z \neq x$. We denote this surjection by π_x . It's also easy to see from the definition that there is an algebra morphism $\iota_x : H_x \rightarrow HC_0(\beta)$ such that $\iota_x(a_{ij}^0) = a_{ij}^0, \iota_x(a_{ij}^x) = a_{ij}^x$, and thus $\pi_x \iota_x = Id$. Therefore, we conclude that for all $x \in \mathbb{Z}$, H_x is a subalgebra of $HC_0(\beta)$, and that

$HC_0(\beta) = H_0 + \sum_{0 \neq x} H_x$, and that for all $m > 0$, $\sum_{|x| \leq m} H_x$ is a proper subalgebra since it does not contain H_y for $|y| > m$.

Next we show that $H_0 \simeq hc_0(\beta)$.

It could be checked that in Equation 2.2, if we set $\Gamma = -1, x = 0$, then Φ_β acting on E_0 is exactly the same as the braid action given in [8] if we make the change of variables as follows: $a_{ij}^0 = \mu a_{ij}$ if $i > j$ and $a_{ij}^0 = a_{ij}$ otherwise. Note that here a_{ij} is the symbol used in [8], but not the $\infty \times \infty$ matrix we defined before. In the language of [8], our a_{ij}^0 is the same as a'_{ij} in that paper.³ Moreover,

$$\begin{aligned} \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,j}^0 &= \Phi_\beta^-(a_{i,n+1}) * a_{n+1,j}^0 = \sum ((\Phi_\beta^L)_{ik} a_{k,n+1}) * a_{n+1,j}^0 \\ &= \sum ((\Phi_\beta^L)_{ik} a_{k,n+1}^0) * a_{n+1,j}^0 = \sum (\Phi_\beta^L)_{ik} a_{k,j}^0. \end{aligned}$$

Then we have

$$A_{ij}^{00} - (\Lambda \Phi_\beta^{-L} A)_{ij}^{00} = a_{ij}^0 - \lambda^{\delta_{i,1}} \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,j}^0 = a_{ij}^0 - \lambda^{\delta_{i,1}} \sum (\Phi_\beta^L)_{ik} a_{k,j}^0,$$

which is exactly the (i, j) -entry of $A - \Lambda \Phi_\beta^L A$ defined in [8]. Similarly, $A_{ij}^{00} - (A \Phi_\beta^{-R} \Lambda^{-1})_{ij}^{00} = a_{ij}^0 - \lambda^{-\delta_{j,1}} a_{i,n+1}^0 * \Phi_\beta^-(a_{n+1,j}^0)$ is the (i, j) -entry of $A - A \Phi_\beta^R \Lambda^{-1}$. Therefore, there is a well-defined isomorphism $H_0 \longrightarrow hc_0(\beta)$ sending a_{ij}^0 to μa_{ij} if $i > j$ and a_{ij} otherwise. \square

Corollary 5. If K is a local knot in $S^1 \times S^2$ with framing l , then $HC_0(K; l)$ is infinitely generated as an R -algebra.

Proof Clear from Proposition 6. \square

We just showed that HC_0 is infinitely generated for local knots. On the other hand, Theorem 7 shows HC_0 is always finitely generated for torus knots. Through some amount of computer calculations, we find that HC_0 is always finitely generated for non-local knots. This motivates us to come up with following conjecture.

Conjecture 1. Let K be a knot in $S^1 \times S^2$ with framing l , then $HC_0(K; l)$ is finitely generated as an R -algebra if and only if K is not local.

4.4. Augmentations. The invariant, HC_0 , could be very difficult to compute for general knots, especially when the number of crossings is large. Thus we will deduce a family of invariants from HC_0 , which are called augmentation numbers and which are relatively easier to

³In the Ng's paper just mentioned, a'_{ij} was defined differently. But we think that was an typo and our argument here is the right way to define it. Also note that the matrix A in that paper has entries a'_{ij} , but not a_{ij} .

compute, at least by computers. The concept of augmentation numbers are introduced in [6] [3] for basically the same reason.

Let $d \geq 2$ be an integer and let $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. Pick three invertible numbers $\lambda_0, \mu_0, \Gamma_0 \in \mathbb{Z}_d$. Then \mathbb{Z}_d can be treated as an R -module, with λ, μ, Γ acting by multiplication by $\lambda_0, \mu_0, \Gamma_0$, respectively. Then $H(\beta; f; d; \lambda_0, \mu_0, \Gamma_0) := HC_0(\beta; f) \otimes_R \mathbb{Z}_d$ is a \mathbb{Z}_d -algebra. Assume $HC_0(\beta; f)$ is finitely generated, then $H(\beta; f; d; \lambda_0, \mu_0, \Gamma_0)$ is a finitely generated \mathbb{Z}_d -algebra, and thus has finitely many algebra morphisms into \mathbb{Z}_d .

Definition 3. Let $\beta \in \mathcal{C}_n, f \in \mathbb{Z}, 2 \leq d \in \mathbb{Z}$ such that $HC_0(\beta; f)$ is finitely generated as an R -algebra, and let $\lambda_0, \mu_0, \Gamma_0 \in \mathbb{Z}_d$ be invertible, then $Aug(\beta; f; d; \lambda_0, \mu_0, \Gamma_0)$ is defined to be the number of algebra morphisms from $H(\beta; f; d; \lambda_0, \mu_0, \Gamma_0)$ to \mathbb{Z}_d .

For example, denote the braid $(\alpha_0 \cdots \alpha_{p-1})^q$ representing the (p, q) -torus knot by $T(p, q)$, then $Aug(T(1, 4); 0; 3; 1, 1, 2) = 4$,
 $Aug(T(1, 5); 0; 3; 1, 1, 2) = 2$, $Aug(T(1, 6); 0; 3; 1, 1, 2) = 4$,
 $Aug(T(1, 4); 0; 5; 1, 1, 3) = 6$, $Aug(T(1, 5); 0; 5; 1, 1, 3) = 3$.

Proposition 7. Set $\lambda = \mu = 1$, then for any $\beta \in \mathcal{C}_n$, there is a $\mathbb{Z}[\Gamma^{\pm 1}]$ -algebra morphism from $HC_0(\beta; f)$ to $\mathbb{Z}[\Gamma^{\pm 1}]$ sending each a_{ij}^x to 2Γ .

Proof Let $t : \mathcal{A}_n \rightarrow R, t(a_{ij}^x) = 2\Gamma$. We first show for $\beta \in \mathcal{C}_n$, $t\Phi_\beta = t$. Clearly, it suffices to prove $t\Phi_{\alpha_k} = t, 0 \leq k \leq n-1$. This can be checked directly from Equations 2.2, 2.3.

Similarly, one can prove $t\Phi_\beta^+ = t\Phi_\beta^- = t$.

We need to show t factors through $\mathcal{I}_{\beta; f; 1, 1}$. Note that now $\Lambda_{\beta; f; 1, 1}$ is the identity matrix.

$$t((\Phi_\beta^{-L} A)_{ij}^{xy}) = t(\Phi_\beta^-(a_{i, n+1}^x) * a_{n+1, j}^y) = t\Phi_\beta^-(a_{i, n+1}^x) = t(a_{i, n+1}^x) = 2\Gamma = t(A_{ij}^{xy}).$$

The other three relations can be verified analogously. \square

Corollary 6. Let $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$ let $\Gamma_0 \in \mathbb{Z}_d$ be invertible, then $Aug(\beta; f; d; 1, 1, \Gamma_0) \geq 1$.

Proof The map t defined in Proposition 7 naturally induces a map from $H(\beta; f; d; 1, 1, \Gamma_0)$ to \mathbb{Z}_d . \square

5. A TOPOLOGICAL INTERPRETATION OF THE KNOT INVARIANT

In this section, we show that the framed knot invariant HC_0 actually has a rather simple interpretation as the framed cord algebra given in Definition 2.2 in [8]. The framed cord algebra is defined for an oriented framed knot K in an oriented 3-manifold M , which is conjectured to be the zero-th relative contact homology of Λ_K in ST^*M . In the same

paper, the author also gave a cord interpretation of the framed cord algebra for knots in S^3 with 0 framing. In the following, we modify the cord interpretation so that it adapts to the knots with any framing, and prove that the modified version is equivalent to the framed cord algebra. Then we show that the knot invariant HC_0 coincides with the framed cord algebra.

Definition 4. Suppose M is an oriented 3-manifold, and K an oriented framed knot in M with l, m the homotopy classes of the longitude and the meridian of K in $\pi_1(M \setminus K)$. Fix a point $*$ on l .

1). A cord in M relative to (K, l) is a continuous map $\gamma : [0, 1] \rightarrow M \setminus K$, such that $\gamma(0), \gamma(1) \in l$ and $\gamma^{-1}(*) = \emptyset$. Two cords γ_1, γ_2 are said to be equivalent if they are homotopic relative to $l \setminus \{*\}$. Informally speaking, one can slide a cord γ along l , so long as not to pass through the point $*$.

2). Let R be the ring $\mathbb{Z}[\lambda^\pm, \mu^\pm, \Gamma^\pm]$. The framed cord algebra, $A(K, l; R)$, is defined as the algebra over R freely generated by the equivalent classes of cords, modulo the ideal generated by the relations given in Figure 14.

$$\begin{aligned}
1) \quad & \text{Diagram: A dashed line } l \text{ with a loop attached.} = (1 + \mu)\Gamma \\
2) \quad & \text{Diagram: A dashed line } l \text{ with a point } * \text{ and a cord passing through it.} = \lambda \text{ Diagram: A dashed line } l \text{ with a point } * \text{ and a cord passing through it.} = \lambda^{-1} \text{ Diagram: A dashed line } l \text{ with a point } * \text{ and a cord passing through it.} \\
3) \quad & \text{Diagram: A dashed line } l \text{ with a knot } K \text{ passing through it.} + \text{Diagram: A dashed line } l \text{ with a knot } K \text{ passing through it.} = \frac{1}{\Gamma} \text{Diagram: A dashed line } l \text{ with a knot } K \text{ passing through it.} \otimes \text{Diagram: A dashed line } l \text{ with a knot } K \text{ passing through it.}
\end{aligned}$$

FIGURE 14. Skein relation

In Figure 14, the dashed line stands for the curve representing l , and the cord is represented by the solid line transversal to l while the knot is drawn as the solid line parallel to l . In the third relation, the diagrams are understood to depict some local neighborhood outside of which the diagrams agree, and the meridian m is assumed to rotate around K counter clock-wise under the current projection.

Now we prove that the framed cord algebra is isomorphic to the one defined in [8]. For the readers convenience, we first recall the definition of framed cord algebra there.

Definition 5. [8] Let $K \subset M$ be an oriented framed knot in an oriented 3-manifold M , and let l, m denote the homotopy classes of the longitude and meridian of K in $\pi_1(M \setminus K)$. The framed cord algebra, $\tilde{A}(K, l; M)$, of K is the algebra over R freely generated by the elements of $\pi_1(M \setminus K)$, modulo the ideal generated by the relations

- 1). $[e] = (1 + \mu)\Gamma$;
- 2). $[\gamma l] = [l\gamma] = \lambda[\gamma]$ for $\gamma \in \pi_1(M \setminus K)$;
- 3). $[\gamma_1 \gamma_2] + [\gamma_1 m \gamma_2] = \frac{1}{f}[\gamma_1][\gamma_2]$, for $\gamma_1, \gamma_2 \in \pi_1(M \setminus K)$.

Remark 11. 1). If we set $\gamma_1 = \gamma, \gamma_2 = e$, then from the first and the third relation, we can derive the relation $[\gamma m] = \mu[\gamma]$. Similarly, we have $[m\gamma] = \mu[\gamma]$.

2). If $[l'] = [l][m]^f$, then $\tilde{A}(K, l'; M)$ can be obtained from $\tilde{A}(K, l; M)$ by replacing λ by $\lambda\mu^{-f}$.

Clearly, the framed cord algebra does not depend on the choice of the base point in defining $\pi_1(M \setminus K)$.

Proposition 8. The framed cord algebras defined in Definition 4 and Definition 5 coincide, namely, $A(K, l; M) \simeq \tilde{A}(K, l; M)$ for an oriented knot K with framing (longitude) given by l in the manifold M .

Proof Assume the base point p is on the curve l , different from the point $*$. And for a point $z \in l$, let τ_z be the sub arc of l connecting p to z not passing the point $*$. Clearly, an element of $\pi_1(M \setminus K)$ is automatically an equivalence class of cords. Also it's easy to see that the three relations in defining $\tilde{A}(K, l; M)$ turn into the three relations defining $A(K, l; M)$, respectively. Conversely, for a cord γ , let $\tilde{\gamma} = \tau_{\gamma(0)} * \gamma * \bar{\tau}_{\gamma(1)}$. Then $\tilde{\gamma}$ is an element of $\pi_1(M \setminus K)$, and this map also preserves the defining relations. \square

Theorem 8. Let $\beta \in \mathcal{C}_n$ be a braid whose closure is a knot in $S^1 \times S^2$, and let l, m be the homotopy classes of the longitude and the meridian of $\hat{\beta}$ in $\pi_1(S^1 \times S^2 \setminus \hat{\beta})$, such that $[l] = [\hat{\beta}'][m]^f$, where $\hat{\beta}'$ is a parallel copy diagram of $\hat{\beta}$, and $f \in \mathbb{Z}$ is an integer. Then we have $HC_0(\beta; f) \simeq A(\hat{\beta}, l; S^1 \times S^2)$.

Proof By the second part of Remark 11 and the properties of $HC_0(\beta; f)$, it suffices to prove the theorem for $f = 0$, namely $[l] = [\hat{\beta}']$. Set $\Lambda = \Lambda_{\beta; 0; 1, 1}$.

Let $X = D_n \times [0, 1] / \{(x, 0) \sim (x, 1), x \in D_n\}$. Present β as a braid diagram inside X . See Figure 3. Assume β intersect D_n in p_1, \dots, p_n . Take a parallel copy diagram β' of β , such that β' intersects with D_n in the points q_1, \dots, q_n . Also choose some point on $\hat{\beta}'$ right above q_1 as the point $*$.

It's clear that any cord in $S^1 \times S^2$ relative to $(\hat{\beta}, \hat{\beta}')$ can be homotoped to inside X . Then we slide the cord γ along $\hat{\beta}'$ and whenever the cord passes the point $*$, we will multiply $\lambda(\lambda^{-1})$ to it according to the second relation in Figure 14. Finally the cord is slid into $D_n \times \{0\}$. we denote the resulting curve by $\tilde{\gamma}$, which is an element in \mathcal{Q}_n .

We define the map $\varphi : A(\hat{\beta}, l; S^1 \times S^2) \longrightarrow HC_0(\beta; 0)$ by sending any cord γ to $\lambda^s \tilde{\gamma}$, where λ^s is the scalar gathered on the way to transit γ into $\tilde{\gamma}$, as stated in the above paragraph. There are several points where we need to check the map is well-defined.

Step 1: The projection of γ to $D_n \times \{0\}$ is not unique, and different projections differ by actions of Φ_β . So we need to show for any $\gamma \in \mathcal{Q}_n$ from q_i to q_j , we have $\gamma = \lambda^{\delta_{i,1}} \Phi_\beta(\gamma) \lambda^{-\delta_{j,1}}$ in $HC_0(\beta; 0)$. Since γ can be written as a sum of monomials of the form $a_{i,i_1}^{x_1} a_{i_1,i_2}^{x_2} \cdots a_{i_{k-1},j}^{x_k}$, by Corollary 4, $\gamma - \lambda^{\delta_{i,1}} \Phi_\beta(\gamma) \lambda^{-\delta_{j,1}}$ is contained in $\mathcal{I}_{\beta;0;1,1}$ and thus 0 in $HC_0(\beta; 0)$.

Step 2: In $S^1 \times S^2$, the cords have more flexibilities to be homotoped than in X . Precisely, there are two more type of flexibilities. Let γ_1, γ_2 be two curves in D_n such that $\gamma_1(1) = \gamma_2(0) = z_1, \gamma_1(0) = q_i, \gamma_2(1) = q_j$, and let δ be the loop $\{z_1\} \times S^1$, then it's clear that $\gamma_1 * \gamma_2, \gamma_1 * \delta * \gamma_2$ are equivalent cords in $S^1 \times S^2$ but not in X . If we project $\gamma_1 * \delta * \gamma_2$ to $D_n \times \{0\}$, then we get $\lambda^{\delta_{i,1}} \Phi_\beta^-(\gamma_1) * \gamma_2$ or $\gamma_1 * \Phi_\beta^-(\gamma_2) \lambda^{-\delta_{j,1}}$. These are guaranteed by the relations $A - \Lambda \Phi_\beta^{-L} A, A - A \Phi_\beta^{-R} \Lambda^{-1}$. See Part (2) of Remark 7.

Similarly, in the above argument, if we replace “ z_1 ” by “ z_0 ”, then we get the relations $A - \Lambda \Phi_\beta^{+L} A, A - A \Phi_\beta^{+R} \Lambda^{-1}$.

Step 3: The first and the third relation in Figure 14 that defines $A(\hat{\beta}, l; S^1 \times S^2)$ are apparently mapped to the two “skein” relations that define \tilde{A}_n . And the second relation in the same figure is also preserved by the map.

The above three steps showed that φ is well-defined. It's also easy to prove it's a bijection.

□

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